Systematic Approach for Portmanteau Tests in View of Whittle Likelihood Ratio

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SUMMARY

Box and Pierce (1970) proposed a test statistic T_{BP} which is the squared sum of m sample autocorrelations of the estimated residual process of autoregressive-moving average model of order (p,q). T_{BP} is called the classical portmanteau test. Under the null hypothesis that the autoregressive-moving average model of order (p,q) is adequate, they suggested that the distribution of T_{BP} is approximated by chi-square distribution with (m-p-q) degrees of freedom, "if m is moderately large". This paper shows that T_{BP} is understood as a special form of Whittle likelihood ratio test T_{PW} for autoregressive-moving average spectral density with m-dependent residual process. Then, it is shown that, for any finite m, T_{PW} does not converge to chi-square distribution with (m-p-q) degrees of freedom in distribution, and that, if we assume Bloomfield's exponential spectral density T_{PW} is asymptotically chi-square distributed for any finite m. From this observation we propose a modified $T^{\dagger}_{PW}$ which is asymptotically chi-square distributed. In view of likelihood ratio, we also mention the asymptotics of a natural Whittle likelihood ratio test T_{WLR} which is always asymptotically chi-square distributed. Its local power is also evaluated. Numerical studies illuminate interesting features of T_{PW}, $T^{\dagger}_{PW}$, and T_{WLR}. Because many versions of the portmanteau test have been proposed, and been used in variety of fields, our systematic approach for portmanteau tests and proposal of tests will give another view and useful applications.

Some Key words: ARMA model; Exponential spectral model; LAN; local power; Portmanteau test; Time series model checking; Whittle likelihood.

1. Introduction

In time series model building, it is usual to verify the adequacy of a fitted model by computing residual autocorrelations. For this Box and Pierce (1970) proposed a test statistic

$$T_{BP} = n \sum_{k=1}^{m} \hat{r}_k^2, \tag{1·1}$$

where $\hat{r}_k$ is the sample autocorrelation of lag k of the estimated residual process. Here n is the sample size, and T_{BP} is called the portmanteau test statistic. Under the null hypothesis that the ARMA(p,q) model is adequate, Box and Pierce (1970) suggested that the distribution of T_{BP} is approximated by χ^2_{m-p-q}, "if m and n are moderately large". However, Davies et al. (1977) claimed that the χ^2_{m-p-q} -approximation is not adequate, i.e., showed that, even for moderately large n and $m = 20$, the true significance levels are likely to be much lower than predicted by asymptotic theory. Ljung and Box (1978) proposed an improved version of T_{BP}:

$$T_{LB} = n(n+2)\sum_{k=1}^{m}(n-k)^{-1}\hat{r}_k{}^2, \tag{1·2}$$

which is called the Ljung-Box test statistic. However, Ansley and Newbold (1979) reported that the asymptotic significance levels by T_{LB} yield a serious understatement. Peña and Rodríguez (2002) proposed a new portmanteau test for time series which is more powerful than Ljung and Box test. For diagnostic checking in ARMA models with non-independent innovations, Francq, Roy and Zakoïan (2005) showed that portmanteau tests can perform poorly in this framework. Various modified versions of portmanteau test can be found in e.g., Lobato et al (2001), Hipel and McLeod (2005), Li (2004), Arranz (2005) and Katayama (2007, 2008).

In many application fields, portmanteau tests, especially, T_{BP} and T_{LB}, have been widely used. It is very important to develop the systematic asymptotic theory which grasps the portmanteau tests. This paper elucidates that the portmanteau tests are essentially equivalent to a special form of Whittle likelihood ratio T_{PW} for the spectral density $f_{(\theta_1,\theta_2)}(\lambda)$ of (2·3) in Section 2, which tests whether the residual correlation parameter θ_2 satisfies $H : \theta_2 = 0$ or $A : \theta_2 \neq 0$. Then, it is shown that, under H, for any finite $m = dim\ \theta_2$, $T_{PW} \nrightarrow \chi^2_{m-p-q}$ in distribution as $n \to \infty$. This result is caused by the fact that T_{PW} uses the Whittle estimator $\hat{\theta}_1$ for the model $f_{(\theta_1,0)(\lambda)}$ and that $\hat{\theta}_2(\hat{\theta}_1)$ for the estimated model $f_{(\hat{\theta}_1,\theta_2)}(\lambda)$. As an auxiliary result we show that, if the time series structure has Bloomfield's exponential spectral model, then, for any finite m, $T_{PW} \to \chi^2_{m-dim\ \theta_1}$, in distribution under H. Also we propose a modified $T^{\dagger}_{PW}$ which is asymptotically chi-square distributed.

In view of likelihood ratio we mention the asymptotics of a natural Whittle likelihood ratio test T_{WLR} which is based on $\hat{\theta}_1$ and $(\tilde{\theta}_1, \tilde{\theta}_2)$ which is the Whittle estimator for the model $f_{(\theta_1,\theta_2)}(\lambda)$. Then it is shown (i) $T_{WLR} \to \chi^2_m$ in distribution under H, and (ii) $T_{WLR} \to$ a noncentral χ^2-distribution in distribution under a sequence of contiguous alternatives $A_n : \theta_2 = h/\sqrt{n}$. Numerical studies for T_{PW}, $T^{\dagger}_{PW}$ and T_{WLR} are provided. They illuminate an interesting feature of them. Since the portmanteau tests are important benchmark statistics, our systematic studies for them give another view.

2. Interpretation of portmanteau test as a special Whittle likelihood ratio

In this section we show that portmanteau tests proposed by Box and Pierce (1970), Ljung and Box (1978), etc., are some special forms of Whittle likelihood ratio test for spectra of concerned stationary processes.

Suppose that $\{X_t\}$ is generated by

$$\sum_{j=0}^{p} \alpha_j X_{t-j} = \sum_{j=0}^{q} \beta_j u_{t-j}, \quad (\alpha_0 = \beta_0 = 1, \alpha_p \neq 0,\ \beta_q \neq 0), \tag{2·1}$$

where $\{u_t\}$ is a sequence of independent and identically distributed $(0, \sigma_u^2)$ random variables with fourth-order cumulant κ_4. Here $\alpha(z) \equiv \sum_{j=0}^{p} \alpha_j z^j$ and $\beta(z) \equiv \sum_{j=0}^{q} \beta_j z^j$ are assumed to satisfy $\alpha(z) \neq 0$ and $\beta(z) \neq 0$ on $\mathbf{D} = \{z \in \mathbf{C} : |z| \leq 1\}$ and the equations $\alpha(z) = 0$ and $\beta(z) = 0$ have no common roots. Then $\{X_t\}$ is stationary with spectral density

$$\begin{aligned} f_{\theta_1}(\lambda) &= \frac{|\sum_{j=0}^{q} \beta_j e^{ij\lambda}|^2}{|\sum_{j=0}^{p} \alpha_j e^{ij\lambda}|^2} \frac{\sigma_u^2}{2\pi} \\ &= g_{\theta_1}(\lambda) \cdot \frac{\sigma_u^2}{2\pi}, \quad (say), \end{aligned} \tag{2·2}$$

where $\theta_1 = (\theta_{1,1}, \cdots, \theta_{1,p+q})' \equiv (\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q)'$. Letting $\theta = (\theta_1', \theta_2')'$, where $\theta_2 = (\theta_{2,1}, \cdots, \theta_{2,m})'$, we introduce the following spectral density

$$\begin{aligned} f_\theta(\lambda) \equiv f_{(\theta_1,\theta_2)}(\lambda) &= \frac{|\sum_{j=0}^{q} \beta_j e^{ij\lambda}|^2}{|\sum_{j=0}^{p} \alpha_j e^{ij\lambda}|^2} \cdot \frac{\sigma_u^2}{2\pi} \left\{ \sum_{j=-m}^{m} \theta_{2,j} e^{-ij\lambda} \right\} \\ &= g_{\theta_1}(\lambda) \cdot \frac{\sigma_u^2}{2\pi} \left\{ \sum_{j=-m}^{m} \theta_{2,j} e^{-ij\lambda} \right\}, \end{aligned} \tag{2·3}$$

where $\theta_{2,0} \equiv 1$, $\theta_{2,-j} \equiv \theta_{2,j}$. It is seen that $f_{(\theta_1,\theta_2)}(\lambda)$ is the spectral density of $\{X_t\}$ in (2·1) if $\{u_t\}$ is an m-dependent sequence with autocovariance $\{\theta_{2,j}\}$, and that $f_{\theta_1}(\lambda)$ in (2·2) is the spectral density when $\{u_t\}$ is independent and identically distributed with $Eu_t = 0$ and $Eu_t^2 = \sigma_u^2$.

Consider the problem of testing

$$H : \theta_2 = 0 \qquad against \qquad A : \theta_2 \neq 0, \tag{2·4}$$

which will lead to the problem of portmanteau test. This is rewritten as $H : u_t = \epsilon_t$ against $A : u_t = \sum_{j=-m}^{m} \theta_{2,j} \epsilon_{t-j}$ where $\{\epsilon_t\}$ is a sequence of i.i.d.$(0, \sigma^2)$.

Let $\vec{X}_n = (X_1, \cdots, X_n)'$ be an observed stretch from (2·1), and write the periodogram as

$$I_n(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} X_t e^{it\lambda} \right|^2, \quad \lambda \in [-\pi, \pi]. \tag{2·5}$$

Although we do not assume Gaussianity of $\{X_t\}$, if $\{X_t\}$ were Gaussian, the log-likelihood based on $\vec{X}_n$ would be approximated by

$$-\frac{n}{4\pi} \int_{-\pi}^{\pi} \{\log f_\theta(\lambda) + \frac{I_n(\lambda)}{f_\theta(\lambda)}\} d\lambda, \tag{2·6}$$

(e.g., Dzhaparidze (1986, p.52), Taniguchi and Kakizawa (2000, section 3.1). Hence we construct a test statistic by use of

$$D(f_\theta, I_n) = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\{\log f_\theta(\lambda) + \frac{I_n(\lambda)}{f_\theta(\lambda)}\}d\lambda. \tag{2·7}$$

For this we define estimators $\hat{\theta_1}$ and $\hat{\theta_2}$ of θ_1 and θ_2, respectively, as follows:

$$\hat{\theta_1} \equiv \arg\max_{\theta_1} D(f_{(\theta_1,0)}, I_n), \tag{2·8}$$

$$\hat{\theta_2}(\hat{\theta}_1) \equiv \arg\max_{\theta_2} D(f_{(\hat{\theta_1},\theta_2)}, I_n), \tag{2·9}$$

where 0 in (2·8) is the m-dimensional zero vector. Here it should be noted that $\hat{\theta_2}(\hat{\theta}_1)$ is a function of $\hat{\theta_1}$. For the testing problem (2·4), we introduce a sort of Whittle likelihood ratio test

$$T_{PW} = 2n[D(f_{(\hat{\theta_1},\hat{\theta_2}(\hat{\theta_1}))}, I_n) - D(f_{(\hat{\theta_1},0)}, I_n)] \tag{2·10}$$

We call T_{PW} a portmanteau test of Whittle type.

Then we have the following theorem.

Theorem 1. Under $H : \theta_2 = 0$ in (2·3), the following statements hold true.

(i) $T_{PW} - T_{BP} \xrightarrow{P} 0$ and $T_{PW} - T_{LB} \xrightarrow{P} 0$ as $n \to \infty$.

(ii) For any fixed $m = \dim\theta_2$, the asymptotic distribution of T_{PW} does not converge to χ^2_{m-p-q} as $n \to \infty$.

We place all the proofs of theorems in Section 5.

Remark 1. In the literature of portmanteau tests, it is claimed that the distribution of portmanteau tests converges to χ^2_{m-p-q} as $n \to \infty$ if m is "sufficient large". Katayama (2008) discussed convergence of T_{BP} and T_{LB} to χ^2_{m-p-q} if $m \to \infty$. But it should be noted that, "if m is finite, it does not converge to χ^2_{m-p-q}" even if $n \to \infty$. In fact, for AR(1) model with coefficient α_1, Ljung (1986) showed that $T_{BP} \sim \chi^2_{m-1} + \alpha_1^{2m}\chi^2_1$, asymptotically, which affirms these statements. There are many works which say that the χ^2_{m-p-q} approximations for portmanteau tests are not adequate (e.g., Davies et al. (1977)). In view of our theorem, the results seem natural.

Portmanteau tests have been used for ARMA models.

Let $\{X_t\}$ be generated by

$$X_t = \sum_{j=0}^{\infty} a_j(\theta_1)u_{t-j}, \tag{2·11}$$

where $\theta_1 = (\theta_{1,1}, \cdots, \theta_{1,r})'$ and $\{u_t\}$ is a sequence of random variables with $Eu_t = 0$, $Eu_t^2 = \sigma_u^2$ and fourth-order cumulant κ_4. We assume that $a_j(\theta_1)$'s are continuously twice differentiable with respect to θ_1, and satisfy

$$\sum_{j=0}^{\infty} a_j(\theta_1)^2 < \infty. \tag{2·12}$$

If $\{u_t\}$ is uncorrelated, then $\{X_t\}$ has the spectral density

$$\begin{aligned} f_{\theta_1}(\lambda) &= \left| \sum_{j=0}^{\infty} a_j(\theta_1) e^{ij\lambda} \right|^2 \cdot \frac{\sigma_u^2}{2\pi} \\ &= g_{\theta_1}(\lambda) \cdot \frac{\sigma_u^2}{2\pi}, \quad (say). \end{aligned} \tag{2·13}$$

The spectral density $g_{\theta_1}(\lambda)$ is very general, hence, it includes the ARMA(p,q) of (2·2) as a special case. Letting $\theta = (\theta_1', \theta_2')'$, where $\theta_2 = (\theta_{2,1}, \cdots, \theta_{2,m})'$, we introduce the following spectral density

$$f_\theta(\lambda) \equiv f_{(\theta_1,\theta_2)}(\lambda) = g_{\theta_1}(\lambda) \cdot \frac{\sigma_u^2}{2\pi} \left\{ \sum_{j=-m}^{m} \theta_{2,j} e^{-ij\lambda} \right\}, \tag{2·14}$$

where $\theta_{2,0} \equiv 1$.

Consider the problem of testing

$$H_G : \theta_2 = 0 \qquad against \qquad A_G : \theta_2 \neq 0, \tag{2·15}$$

which is the generalized form of portmanteau testing problem.

Write

$$F \equiv \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial\theta} \log f_\theta(\lambda) \frac{\partial}{\partial\theta'} \log f_\theta(\lambda) d\lambda = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}.$$

In what follows we assume that F is nonsingular. For our general spectral model (2·14), we have,

Theorem 2. Assume $m = \dim \theta_2 > r = \dim \theta_1$. Then, under $H_G : \theta_2 = 0$ in (2·14), the asymptotic distribution of T_{PW} for (2·14) converges to χ^2_{m-r} as $n \to \infty$ if the matrix $F_{21} F_{11}^{-1} F_{12}$ is idempotent with rank $\{F_{21} F_{11}^{-1} F_{12}\} = r$.

Corollary 2. If $g_{\theta_1}(\lambda)$ in (2·14) is of the form

$$g_{\theta_1}(\lambda) = \frac{\sigma^2}{2\pi} \exp\left[\sum_{j=0}^{r} \theta_{1,j} \cos j\lambda \right], \quad \theta_{1,0} = 1, \tag{2·16}$$

which is called the exponential spectral density (Bloomfield (1973)), then, under H_G : $\theta_2 = 0$,

$$T_{PW} \xrightarrow{d} \chi^2_{m-r} \qquad as\ n \to \infty. \tag{2·17}$$

From Theorems 1 and 2, we observe that the asymptotics of portmanteau type test T_{PW} depend on the time series structure of $\{X_t\}$ strongly.

In the case of ARMA(p,q) model (2·1), Katayama (2008) proposed a modified statistic $T_{PW}^\dagger$ of T_{PW} such that $T_{PW}^\dagger$ is asymptotically χ^2_{m-p-q} distributed under H in (2·4).

For the general spectral model (2·14), such a modification is possible. Since submatrices F_{ij} of the Fisher information matrix depend on the unknown parameter $\theta = (\theta_1', \theta_2')'$, i.e, $F_{ij} = F_{ij}(\theta)$, we estimate F_{ij} by $\tilde{F}_{ij} \equiv F_{ij}(\tilde{\theta})$ where $\tilde{\theta} \equiv (\hat{\theta}_1', \hat{\theta}_2(\hat{\theta}_1)')'$. Define $\tilde{W} \equiv \tilde{F}_{21}(\tilde{F}_{12}\tilde{F}_{21})^{-1}\tilde{F}_{12}$, and let

$$T_{PW}^{\dagger} \equiv T_{PW} - n\hat{\theta}_2(\hat{\theta}_1)'\tilde{W}\hat{\theta}_2(\hat{\theta}_1). \tag{2·18}$$

Then we have,

Theorem 3. For (2·14), assume $m > r$. Then, under $H_G : \theta_2 = 0$, it holds that

$$T_{PW}^{\dagger} \xrightarrow{d} \chi_{m-r}^2 \qquad (n \to \infty). \tag{2·19}$$

Here, we do not assume that $F_{21}F_{11}^{-1}F_{12}$ is idempotent as in Theorem 2.

3. Power Properties for $T_{PW}^{\dagger}$ and Natural Whittle Likelihood Ratio

This section discusses the local power properties of $T_{PW}^{\dagger}$ and a natural Whittle likelihood ratio test T_{WLR}. In the case of ARMA, Katayama (2007) derived the local power of some portmanteau tests by a Taylor expansion around the hypothesis H. In what follows, for general spectra including ARMA, we derive the local power of $T_{PW}^{\dagger}$ and T_{WLR} by use of the LAN theory and LeCam's third lemma. Although we can use the local asymptotic normality (LAN) result for general non-Gaussian linear processes (Theorem 2.2.1 of Taniguchi and Kakizawa (2000)), to avoid unnecessarily complicated notations and discussion, in what follows, we restrict ourselves to the case when the process (2·11) is Gaussian.

Assumption.

(i) The spectral density $f_\theta(\lambda)$, $\theta = (\theta_1, \theta_2)$, is continuously twice differentiable with respect to θ.

(ii) There exist positive constants c_1 and c_2 such that

$$c_1 \leq f_\theta(\lambda) \leq c_2 \qquad \text{on} \quad [-\pi, \pi]. \tag{3·1}$$

(iii) The Fisher information matrix F is positive definite.

Recall our testing problem:

$$H_G : \theta_2 = 0 \quad \text{against} \quad A_G : \theta_2 \neq 0. \tag{3·2}$$

We evaluate the local power of $T_{PW}^{\dagger}$ under a local alternative

$$A_G^{(n)} : \theta_2 = \frac{1}{\sqrt{n}}h, \tag{3·3}$$

where h is a fixed m-dimensional vector.

Theorem 4. Suppose that Assumption holds. Then, under $A_G^{(n)}$,

$$T_{PW}^{\dagger} \xrightarrow{d} \chi_{m-r}^2\{h'Ch\} \qquad as\ n \to \infty, \tag{3·4}$$

where $C = l_{m\times m} - F_{21}(F_{12}F_{21})^{-1}F_{12}$, and $\chi^2_{m-r}\{h'Ch\}$ is a noncentral χ^2 random variable with $(m-r)$ degrees of freedom and noncentrality $h'Ch$.

For the testing problem (3·2), we are led to think of a natural Whittle likelihood ratio test.

Define

$$(\tilde{\theta}_1, \tilde{\theta}_2) \equiv \arg\max_{(\theta_1,\theta_2)} D(f_{(\theta_1,\theta_2)}, I_n). \tag{3·5}$$

Here we should note that the estimator $(\hat{\theta}_1, \hat{\theta}_2(\hat{\theta}_1))$ defined by (2·8) and (2·9) is essentially different from $(\tilde{\theta}_1, \tilde{\theta}_2)$. Based on the estimator $(\tilde{\theta}_1, \tilde{\theta}_2)$ we can construct the following Whittle likelihood ratio test

$$T_{WLR} \equiv 2n[D(f_{(\tilde{\theta}_1,\tilde{\theta}_2)}, I_n) - D(f_{(\hat{\theta}_1,0)}, I_n)] \tag{3·6}$$

for the testing problem H_G against A_G.

Write $D(f_\theta, I_n)$ in (2·7) as $l(\theta_1, \theta_2)$. For the problem of testing H: $\theta_2 = 0$ v.s. A: $\theta_2 \neq 0$, Newbold (1980) and Li (2004, p.14) used the Lagrange multiplier test

$$LM = n\left\{\frac{\partial}{\partial\theta} l(\hat{\theta}_1, \mathbf{0})\right\}' \left[E\left\{-\frac{\partial^2}{\partial\theta\partial\theta'} l(\hat{\theta}_1, \mathbf{0})\right\}\right]^{-1} \left\{\frac{\partial}{\partial\theta} l(\hat{\theta}_1, \mathbf{0})\right\}. \tag{3·7}$$

Newbold (1980) showed that LM test of ARMA(p,q) against ARMA(p+k,q) is asymptotically equivalent to a standardized quadratic form of k residual autocorrelations.

For general spectral densities (2·13) and (2·14) which include ARMA spectra, we have the following unified results.

Theorem 5. Suppose that Assumption holds, and $m = \dim\theta_2$. Then, for any fixed m, the following statements hold true.

(i) Under H_G,

$$T_{WLR} \xrightarrow{d} \chi^2_m, \qquad as\ n \to \infty. \tag{3·8}$$

(ii) Under H_G, T_{WLR} is asymptotically equivalent to the LM test (3·7).

(iii) Under $A_G^{(n)}$,

$$T_{WLR} \xrightarrow{d} \chi^2_m(h'F_{22\cdot1}h) \qquad as \quad n \to \infty, \tag{3·9}$$

where $F_{22\cdot1} = F_{22} - F_{21}F_{11}^{-1}F_{12}$, and $\chi^2_m(h'F_{22\cdot1}h)$ is a noncentral χ^2 random variable with m degrees of freedom and noncentrality parameter $h'F_{22\cdot1}h$.

In the next section we will provide numerical results for T_{PW}, $T^{\dagger}_{PW}$ and T_{WLR}.

4. Numerical studies for T_{WLR} and $T^{\dagger}_{PW}$

In this section, we give numerical studies of our test statistic T_{WLR} and $T^{\dagger}_{PW}$. In Example 4.1, we compare the finite-sample significance levels of T_{WLR}, $T^{\dagger}_{PW}$ with another famous portmanteau test T_{LB} under MA(1) process. In Example 4.2, under AR(1) process the finite-sample significance levels of T_{WLR}, $T^{\dagger}_{PW}$ and T_{LB} are examined. Then it can be seen that T_{WLR} and $T^{\dagger}_{PW}$ are more accurate than T_{LB}. In Example 4.3, we analyse the local powers of T_{WLR} and $T^{\dagger}_{PW}$ under local alternative and we can observe some interesting power properties. In Examples 4.1 and 4.2, the simulations are besed on 5000 realizations and $n = 200$.

Example 4.1. Let $\{X_t\}$ be the MA(1) process

$$X_t = u_t + \beta u_{t-1} \tag{4·1}$$

where u_t's are independent and identically distributed as $N(0, 1)$. In Table 1, we report the 5% empirical significance levels of T_{WLR} for $m = 1$, $T^{\dagger}_{PW}$ for $m = 2$ and T_{LB} for $m = 20$. The parameter values are chosen as $0.1 \le \beta \le 0.9$.

Table 1 is about here.

From Table 1, we can see that the empirical significance levels of T_{WLR}, $T^{\dagger}_{PW}$ are closer to the assigned value than those of T_{LB}.

Example 4.2. Let $\{X_t\}$ be the AR(1) process

$$X_t + \alpha X_{t-1} = u_t \tag{4·2}$$

where u_t's are independent and identically distributed as $N(0, 1)$. In Table 2, the 5% empirical significance levels of T_{WLR} for $m = 1$, $T^{\dagger}_{PW}$ for $m = 2$ and T_{LB} for $m = 20$ are reported. The parameter values are chosen as $0.1 \le \alpha \le 0.9$.

Table 2 is about here.

From Table 2, we can see that T_{WLR}, $T^{\dagger}_{PW}$ are better than T_{LB}.

Example 4.3. Let $\{X_t\}$ be the ARMA(1,1) process

$$X_t + \alpha_1 X_{t-1} = u_t + \beta_1 u_{t-1} \tag{4·3}$$

where $\{u_t\}$ is an m-dependent sequence with mean 0, variance 1 and its autocovariance functions are $\theta_2 = \frac{2}{\sqrt{n}}(1, 1, \cdots, 1)$. The parameter values are taken as $\alpha_1 = 0.2, 0.4, 0.6, 0.8$ and $\beta_1 = 0.1, 0.3, 0.5, 0.7, 0.9$. From Theorem 5, T_{WLR} converges to $\chi^2_m(h'F_{22\cdot1}h)$ as $n \to \infty$. In Tables 3 and 4, the theoretical local powers for an 5% level test of T_{WLR} are reported for $m = 5, 10$ respectively.

Tables 3 and 4 are about here.

From Tables 3 and 4 we can see that the theoretical local power of T_{WLR} increases as the parameter values α_1 and β_1 become large.

From Theorem 4, $T^{\dagger}_{PW}$ converges to $\chi^2_{m-r}\{h'Ch\}$ as $n \to \infty$. In Tables 5 and 6, we report the theoretical local powers of $T^{\dagger}_{PW}$ for an 5% level test for $m = 5, 10$ respectively.

Tables 5 and 6 are about here.

From Tables 5 and 6 it may be noted that local power of $T^{\dagger}_{PW}$ increases as the parameter values α_1 and β_1 become large.

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References

[1] Ansley, C. F. & Newbold, P. (1979) On the finite sample distribution of residual autocorrelations in autoregressive-moving average models. Biometrika 66, 547-553.

[2] Arranz, M. A. (2005) Portmanteau test statistics in time series. Tol-Project. http://www.tol-project.org:8081/docs/ndmtest.pdf

[3] Bloomfield, P. (1973) An exponential model for the spectrum of a scalar time series. Biometrika 60, 217-226.

[4] Box, G. E. P. & Pierce, D. A. (1970) Distribution of residual autocorrelations in autoregressive-integrated moving average time series models. J. Amer. Statist. Assoc. 65, 1509-1526.

[5] Brockwell, P. J. & Davis, R. A. (1991) *Time Series: Theory and Methods, 2nd ed.* New York: Springer-Verlag.

[6] Davies, N., Triggs, C. M. & Newbold, P. (1977) Significance levels of the Box-Pierce portmanteau statistic in finite samples. Biometrika 64, 517-522.

[7] Dzhaparidze, K. (1986) *Parameter Estimation and Hypothesis Testing in Spectral Analysis of Stationary Time Series.* New York: Springer-Verlag.

[8] Francq, C., Roy, R. & Zakoian, J. (2005) Diagnostic checking in ARMA models with uncorrelated errors. J. Amer. Statist. Assoc. 100, 532-544.

[9] Hipel, K. W. & McLeod, A. I. (2005) Time Series Modelling of Water Resources and Environmental Systems. http://www.stats.uwo.ca/faculty/aim/1994Book/default.htm

[10] Katayama, N. (2007) On the Bias of the Portmanteau Statistic, *The Third Symposium on Econometric Theory and Applications* (SETA2007), Available at http://www.bm.ust.hk/SETA2007/acceptedPapers/KATAYAMA_NAOYA.pdf.

[11] Katayama, N. (2008) An improvement of the Portmanteau statistic. J. Time Ser. Anal. 29, 359-370.

[12] Li, W. K. (2004) *Diagnostic checks in Time Series.* New York: Chapman & Hall/CRC.

[13] Ljung, G. M. (1986) Diagnostic testing of univariate time series models. Biometrika 73, 725-730.

[14] Ljung, G. M. & Box, G. E. P. (1978) On a measure of lack of fit in time series models. Biometrika 65, 297-303.

[15] Lobato, I., Nankervis, J. C. & Savin, N. E. (2001) Testing for autocorrelation using a modified Box-Pierce Q test. Internat. Econom. Rev. 42, 187-205.

[16] Newbold, P. (1980) The equivalence of two tests of time series model adequacy. Biometrika 67, 463-465.

[17] Peña, D. & Rodríguez, J. (2002) A powerful portmanteau test of lack of fit for time series. J. Amer. Statist. Assoc. 97, 601-610.

[18] Taniguchi, M. & Kakizawa, Y. (2000) *Asymptotic Theory of Statistical Inference for Time Series.* New York: Springer-Verlag.

β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
T_{WLR}	0.036	0.043	0.043	0.059	0.077	0.074	0.068	0.057	0.052
$T^{\dagger}_{PW}$	0.048	0.047	0.048	0.048	0.05	0.05	0.049	0.052	0.05
T_{LB}	0.072	0.073	0.067	0.076	0.077	0.074	0.083	0.123	0.276

Table 1: Empirical significance levels of T_{WLR}, $T^{\dagger}_{PW}$ and T_{LB} in Example 4.1.

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
T_{WLR}	0.050	0.059	0.066	0.059	0.056	0.055	0.055	0.045	0.041
$T^{\dagger}_{PW}$	0.054	0.053	0.049	0.048	0.052	0.044	0.048	0.042	0.042
T_{LB}	0.051	0.059	0.061	0.061	0.051	0.058	0.054	0.063	0.06

Table 2: Empirical significance levels of T_{WLR}, $T^{\dagger}_{PW}$ and T_{LB} in Example 4.2.

$\alpha \setminus \beta$	0.1	0.3	0.5	0.7	0.9
0.2	0.840881	0.873037	0.895161	0.909700	0.915722
0.4	0.869604	0.897393	0.915605	0.926447	0.930437
0.6	0.890136	0.913794	0.928195	0.935884	0.938534
0.8	0.902261	0.922552	0.934162	0.940085	0.942624

Table 3: Theoretical local powers of T_{WLR} in the case of $m = 5$ in Example 4.3.

$\alpha \setminus \beta$	0.1	0.3	0.5	0.7	0.9
0.2	0.993079	0.994594	0.995493	0.996032	0.996341
0.4	0.994389	0.995629	0.996357	0.996784	0.997041
0.6	0.995192	0.996255	0.996869	0.997223	0.997470
0.8	0.995648	0.996605	0.997155	0.997489	0.997799

Table 4: Theoretical local powers of T_{WLR} in the case of $m = 10$ in Example 4.3.

$\alpha \setminus \beta$	0.1	0.3	0.5	0.7	0.9
0.2	0.897868	0.921801	0.937450	0.947201	0.951454
0.4	0.919283	0.939051	0.951313	0.958331	0.961202
0.6	0.933815	0.950049	0.959469	0.964473	0.966424
0.8	0.942070	0.955821	0.963442	0.967320	0.968795

Table 5: Theoretical local powers of $T^{\dagger}_{PW}$ in the case of $m = 5$ in Example 4.3.

$\alpha \setminus \beta$	0.1	0.3	0.5	0.7	0.9
0.2	0.995731	0.996724	0.997303	0.997645	0.997798
0.4	0.996591	0.997390	0.997852	0.998118	0.998233
0.6	0.997110	0.997787	0.998172	0.998387	0.998477
0.8	0.997395	0.998000	0.998338	0.998521	0.998596

Table 6: Theoretical local powers of $T^{\dagger}_{PW}$ in the case of $m = 10$ in Example 4.3.