Higher Order Asymptotic Option Valuation for Non-Gaussian Dependent Returns

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Abstract

This paper discusses the option pricing problems using statistical series expansion for the price process of an underlying asset. We derive the Edgeworth expansion for the stock log return via extracting dynamics structure of time series. Using this result, we investigate influences of the non-Gaussianity and the dependency of log return processes for option pricing. Numerical studies show some interesting features of them.

keywords: Black and Scholes model; Edgeworth expansion; Non-Gaussian stationary process; Option pricing

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1. Introduction

Black and Scholes (1973) provided the foundation of modern option pricing theory. Despite its usefulness, however, the Black and Scholes theory entails some inconsistencies. It is well known that the model frequently misprices deep in-the-money and deep out-of-the-money options. This result is generally attributed to the unrealistic assumptions used to derive the model. In particular, the Black and Scholes model assumes that stock prices follow a geometric Brownian motion with a constant volatility under an equivalent martingale measure.

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In order to avoid this drawback, Jarrow and Rudd (1982) proposed a semi-parametric option pricing model to account for non-normal skewness and kurtosis in stock returns. This approach aims to approximate the risk-neutral density by a statistical series expansion. Jarrow and Rudd (1982) approximated the density of the state price by an Edgeworth series expansion involving the log-normal density. Corrado and Su (1996a) implemented Jarrow and Rudd’s formula to price options. Corrado and Su (1996b and 1997) considered Gram-Charlier expansions for the stock log return rather than the stock price itself. Rubinstein (1998) used the Edgeworth expansion for the stock log return. Jurczenko et al. (2002) compared these different multi-moment approximate option pricing models. Also they investigated in particular the conditions that ensure the martingale restriction.

As in Kariya (1993) and Kariya and Liu (2003), the time series structure of return series does not always admit a measure which makes the discounted process a martingale. Hence, we will not able to develop an arbitrage pricing theory by forming an equivalent portfolio. In such a case, we often regard the expected value of the present value of a contingent claim as a proxy for pricing maybe with help of a risk neutrality argument. In view of this, Kariya (1993) considered pricing problems with no martingale property and approximated the density of the state price by the Gram-Charlier expansion for the stock log return.

In this paper, we consider option pricing problems by using Kariya’s approach. In Section 2, we derive the Edgeworth expansion for the stock log return via extracting dynamics structure of time series. Using this result, we investigate influences of the non-Gaussianity and the dependency of log return processes for option pricing. Numerical studies illuminate some interesting features of the influences. In Section 3, we give option prices based on the risk neutrality argument. In Section 4, we discuss a consistent estimator of the quantities in our results. Section 5 concludes. The proofs of theorems are relegated to Section 6.

2. Edgeworth expansion of log return

Let \( \{S_t; t \geq 0\} \) be the price process of an underlying security at trading time \( t \). The \( j \)-th period log return \( X_j \) is defined as

\[
\log S_{t_0+j\Delta} - \log S_{t_0+(j-1)\Delta} = \Delta \mu + \Delta^{1/2} X_j, \quad j = 1, 2, \ldots, N,
\]

where \( T_0 \) is present time, \( N = \tau / \Delta \) is the number of unit time intervals of length \( \Delta \) during a period \( \tau = T - T_0 \) and \( T \) is the maturity date. Then the terminal price \( S_T \) of the underlying security is given by

\[
S_T = S_{T_0} \exp \left\{ \tau \mu + \left( \frac{\tau}{N} \right)^{1/2} \sum_{j=1}^{N} X_j \right\}.
\]

Remark 1. In the Black and Scholes option theory the price process is assumed
to be a geometric Brownian motion

\[ S_T = S_{T_0} \exp(\tau \mu + \sigma W_\tau), \]  

(3)

where the process \( \{W_t; t \in \mathbb{R}\} \) is a Wiener process with drift 0 and variance \( t \).

From (3), the log return at discretized time point can be written as

\[ \log S_{t+j\Delta} - \log S_{t+(j-1)\Delta} = \Delta \mu + \Delta^{1/2} \sigma \nu_j, \quad \nu_j \sim iid N(0,1). \]  

(4)

The expression of (1) is motivated from (4).

First, we derive an analytical expression for the density function of \( S_T \). Since from (2) the distribution of \( S_T \) depends on that of \( Z_N = N^{-1/2} \sum_{j=1}^{N} X_j \), we consider the Edgeworth expansion of the density function of \( Z_N \). If we assume that \( X_j \) are independently and identically distributed random variables with mean zero and finite variance, it is easy to give the Edgeworth expansion for \( Z_N \) (the classical Edgeworth expansion).

However, a lot of financial empirical studies show that \( X_j \)'s are not independent. Thus we suppose that \( \{X_j\} \) is a dependent process which satisfies the following assumption.

(A1) \( \{X_t; t \in \mathbb{Z}\} \) is fourth order stationary in the sense that

- (i) \( E(X_t) = 0 \),
- (ii) \( \text{cum}(X_t, X_{t+u}) = c_{X,2}(u) \),
- (iii) \( \text{cum}(X_t, X_{t+u_1}, X_{t+u_2}) = c_{X,3}(u_1, u_2) \),
- (iv) \( \text{cum}(X_t, X_{t+u_1}, X_{t+u_2}, X_{t+u_3}) = c_{X,4}(u_1, u_2, u_3) \).

(A2) The cumulants \( c_{X,k}(u_1, \ldots, u_{k-1}), k = 2, 3, 4 \), satisfy

\[ \sum_{u_1, \ldots, u_{k-1} = -\infty}^{\infty} \left( 1 + |u_j|^{2-k/2} \right)|c_{X,k}(u_1, \ldots, u_{k-1})| < \infty \]

for \( j = 1, \ldots, k-1 \).

(A3) \( J \)-th order (\( J \geq 5 \)) cumulants of \( Z_N \) are all \( O(N^{-j/2+1}) \).

Under (A2), \( \{X_t; t \in \mathbb{Z}\} \) has the \( k \)-th order cumulant spectral density. Let \( f_{X,k} \) be the \( k \)-th order cumulant spectral density evaluated at frequency 0

\[ f_{X,k} = (2\pi)^{-(k-1)} \sum_{u_1, \ldots, u_{k-1} = -\infty}^{\infty} c_{X,k}(u_1, \ldots, u_{k-1}) \]

for \( k = 2, 3, 4 \).

First, we state the following result.
Theorem 1. Suppose that (A1)-(A3) hold. The third order Edgeworth expansion of the density function of $Z = (2\pi f_{X,2})^{-1/2}Z_N$ is given by

$$g(z) = \phi(z) \left\{ 1 + \frac{(2\pi)^{1/2}}{6} N^{-1/2} \frac{f_{X,3}}{(f_{X,2})^{3/2}} H_3(z) - \frac{1}{4\pi} N^{-1} \frac{f_{X,2}}{f_{X,2}} + \frac{\pi}{12} N^{-1} \frac{f_{X,4}}{(f_{X,2})^2} H_4(z) + \frac{\pi^2}{36} N^{-1} \frac{(f_{X,3})^2}{(f_{X,2})^3} H_6(z) \right\} + o(N^{-1}),$$

where $\phi(\cdot)$ is the standard normal density function, $H_k(\cdot)$ is the k-th order Hermite polynomial and

$$f'_{X,2} = \sum_{u=0}^{\infty} \{ u \mid c_{X,2}(u) \}.$$

Many authors have proposed to use different statistical series expansion to price options (see Jarrow and Rudd 1982, Corrado and Su 1996 and 1997, Rubinstein 1998 and Kariya 1993). Here we give the Edgeworth expansion for the stock log return in powers of $N^{-1/2}$.

A European call option can be viewed as a security which pays at time $T$ its holder the amount

$$X_T^* = \max(S_T - K, 0),$$

where $K$ is the exercise or strike price. As in Kariya (1993), we price $X_T^*$ by its discounted expected value;

$$C = \exp(-r\tau) E_{T_0}(X_T^*),$$

where $r$ is the interest rate which is regarded as a constant for the remaining period $\tau$ and $E_{T_0}(\cdot)$ is evaluated at $T_0$. Evaluate (6) based on the density in (5). Then writing

$$d_1 = (\log S_{T_0}/K + \tau \mu + 2\pi \tau f_{X,2})/(2\pi \tau f_{X,2})^{1/2},$$
$$d_2 = d_1 - (2\pi \tau f_{X,2})^{1/2},$$

we obtain the following theorem

Theorem 2. Let $a_1 = \exp(-r\tau)$ and $a_2 = \exp(\tau \mu + \pi \tau f_{X,2})$. Then

$$C = G_0 + \frac{(2\pi)^{1/2}}{6} N^{-1/2} \frac{f_{X,3}}{(f_{X,2})^{3/2}} G_3 - \frac{1}{4\pi} N^{-1} \frac{f_{X,2}}{f_{X,2}} G_2 + \frac{\pi}{12} N^{-1} \frac{f_{X,4}}{(f_{X,2})^2} G_4 + \frac{\pi^2}{36} N^{-1} \frac{(f_{X,3})^2}{(f_{X,2})^3} G_6 + o(N^{-1}),$$

where

$$G_0 = a_1 \{ a_2 S_{T_0} \Phi(d_1) - K \Phi(d_2) \},$$

$$G_k = a_1 a_2 S_{T_0} \left\{ \sum_{j=1}^{k-1} \frac{(2\pi \tau f_{X,2})^{j/2}}{j!} H_{k-j-1}(-d_2) \phi(d_1) + (2\pi \tau f_{X,2})^{k/2} \Phi(d_1) \right\},$$

for $k = 2, 3, 4, 6$, where $\Phi(\cdot)$ is the standard normal distribution function.
From (7) it is seen that the asymptotic expansion of the option price depends on $f_{X,2}, f_{X,3}^2$, $f_{X,3}$ and $f_{X,4}$. Hence, we can see influences of the non-Gaussianity and the dependency of the log return processes for the higher order option valuation.

**Corollary 1.** Write

$$C = G_0 + N^{-1/2}C_{G,2} + N^{-1}C_{G,3} + N^{-1}C_{D,3} + o(N^{-1}),$$

where

$$C_{G,2} = \frac{(2\pi)^{1/2}}{6} \frac{f_{X,3}}{(f_{X,2})^{3/2}} G_3,$$

$$C_{G,3} = \frac{\pi}{12} \frac{f_{X,4}}{(f_{X,2})^2} G_4 + \frac{\pi}{36} \frac{(f_{X,3})^2}{(f_{X,2})^3} G_6,$$

$$C_{D,3} = -\frac{1}{4} f_{X,2}' \frac{G_2}{f_{X,2}}.$$

If $\{X_t; t \in \mathbb{Z}\}$ is independent, then $C_{D,3} = 0$. If $\{X_t; t \in \mathbb{Z}\}$ is a Gaussian process, then $C_{G,2} = C_{G,3} = 0$.

**Example 1.** Suppose that $X_j, j = 1, \ldots, N$, are independently and identically distributed random variables. Let $c_{X,k} = c_{X,k}(0)$, $k = 2, 3, 4$. Note that $f_{X,2}' = 0$ and $f_{X,k} = (2\pi)^{-(k-1)}c_{X,k}$, $k = 2, 3, 4$. The price of a European call option $C_{IID}$ is given by

$$C_{IID} = G_0 + \frac{1}{6} N^{-1/2} \frac{c_{X,3}}{(c_{X,2})^{3/2}} G_3 + \frac{1}{24} N^{-1} \frac{c_{X,4}}{(c_{X,2})^2} G_4 + \frac{1}{72} N^{-1} \frac{(c_{X,3})^2}{(c_{X,2})^3} G_6 + o(N^{-1}),$$

where $G_k$, $k = 0, 3, 4, 6$, are defined in Theorem 2 with $f_{X,2} = (2\pi)^{-1}c_{X,2}$.

If $\mu = r - c_{X,2}/2$, then $a_1 a_2 = 1$ so that $G_0$ equals the Black and Scholes formula.

**Example 2.** In Example 1, suppose that $X_j, j = 1, \ldots, N$, are distributed as $t$-distribution with $\nu$ degrees of freedom. Then, for $\nu > 4$

$$C_t = G_{t,0} + N^{-1}G_{t,3} + o(N^{-1}),$$

where

$$G_{t,0} = a_1 \{a_2 S_{T_0} \Phi(d_1) - K \Phi(d_2)\},$$

$$a_2 = \exp\left\{\frac{\nu}{2} \left(\frac{\mu + \frac{\nu}{2}}{\nu - 2}\right)\right\},$$

$$d_1 = \left(\log S_{T_0}/K + \frac{\nu}{\nu - 2}\right) \left(\frac{\nu}{\nu - 2}\right)^{1/2},$$

$$d_2 = d_1 - \left(\frac{\nu}{\nu - 2}\right)^{1/2}.$$
and
\[ G_{t,3} = \frac{a_1a_2S_{T_0}}{4(\nu - 4)} \left\{ \sum_{j=1}^{3} \left( \frac{\tau \nu}{\nu - 2} \right)^j H_{3-j}(-d_2)\phi(d_1) + \left( \frac{\tau \nu}{\nu - 2} \right)^2 \Phi(d_1) \right\}. \]

In order to show influences of higher order terms, in Figure 1, we plotted \( C_{t,1} = G_{t,0} \) (dotted line) and \( G_{t,3} = G_{t,0} + N^{-1}G_{t,3} \) (solid line) of Example 2 with \( S_{T_0} = K = 100, \tau = 30/365, N = 30 (\Delta = 1/365), r = \mu = 0.05 \) and \( 4 < \nu < 9 \). From this, we observe that \( C_{t,3} \) diverges as \( \nu \to 4 \).

Figure 1 is about here.

Example 3. Let \( \{X_t : t \in \mathbb{Z}\} \) be the ARCH(1) process
\[ X_t = h_t^{1/2} \eta_t \quad \text{and} \quad h_t = \psi_0 + \psi_1X_{t-1}^2, \]
where \( \psi_0 > 0, \psi_1 \geq 0, \{\eta_t : t \in \mathbb{Z}\} \) is a sequence of independently and identically distributed random variables with
\[ E(\eta_t) = 0, \quad E(\eta_t^2) = 1, \]
\[ E(\eta_t^3) = 0, \quad E(\eta_t^4) = m, \quad m > 1, \]
and \( \eta_t \) is independent of \( X_{t-s}, s > 0 \). Then
\[ f_{X,2} = \frac{1}{2\pi} \frac{\psi_0}{1 - \psi_1}, \quad f_{X,3} = 0, \quad f'_{X,2} = 0, \]
\[ f_{X,4} = \frac{1}{(2\pi)^3} \frac{\psi_0^2(m - 3 + 5m\psi_1 - 3\psi_1 + 2m\psi_0^2 - 2m\psi_1^3)}{(1 - \psi_1)^3(1 - m\psi_1^2)}, \]
for \( ma_1^2 < 1 \). Hence,
\[ C_{ARCH(1)} = G_{ARCH(1),0} + N^{-1}G_{ARCH(1),3} + o(N^{-1}), \]
where
\[ G_{ARCH(1),0} = a_1a_2S_{T_0}\Phi(d_1) - a_1K\Phi(d_2), \]
\[ a_2 = \exp \left\{ \tau \mu + \frac{\tau \psi_0}{2(1 - \psi_1)} \right\}, \]
\[ d_1 = \left( \log S_{T_0}/K + \tau \mu + \frac{\tau \psi_0}{1 - \psi_1} \right) / \left( \frac{\tau \psi_0}{1 - \psi_1} \right)^{1/2}, \]
\[ d_2 = d_1 - \left( \frac{\tau \psi_0}{1 - \psi_1} \right)^{1/2}, \]
and
\[ G_{ARCH(1),3} = \frac{a_1a_2S_{T_0}}{24} \frac{m - 3 + 5m\psi_1 - 3\psi_1 + 2m\psi_0^2 - 2m\psi_1^3}{(1 - \psi_1)(1 - m\psi_1^2)} \times \left\{ \sum_{j=1}^{3} \left( \frac{\tau \psi_0}{1 - \psi_1} \right)^j H_{3-j}(-d_2)\phi(d_1) + \left( \frac{\tau \psi_0}{1 - \psi_1} \right)^2 \Phi(d_1) \right\}. \]
In Figure 2, we plotted $C_{ARCH(1),1} = G_{ARCH(1),0}$ (dotted line) and $C_{ARCH(1),3} = G_{ARCH(1),0} + N^{-1}G_{ARCH(1),3}$ (solid line) of Example 3 with $ST_0 = K = 100$, $\tau = 30/365$, $N = 30$ ($\Delta = 1/365$), $r = \mu = 0.05$, $m = 3$, $\psi_0 = 0.5$ and $-1/\sqrt{3} < \psi_1 < 1/\sqrt{3}$. Figure 2 illuminates influences of higher order terms under Gaussian innovations. From this, we can see that $C_{ARCH(1),3}$ diverges as $\psi_1 \to \pm 1/\sqrt{3}$.

In Figure 3, we plotted $C_{ARCH(1),1}$ (dotted line) and $C_{ARCH(1),3}$ (solid line) of Example 3 with $ST_0 = 100$, $K = 95$, $\tau = 30/365$, $N = 30$, $r = \mu = 0.05$, $\psi_0 = 0.5$, $\psi_1 = 0.3$ and $1 < m < 9$. Figure 3 illuminates influences of non-Gaussian innovations. From this, we observe that $C_{ARCH(1),3}$ decreases as $m \to 9$. The first order term $C_{ARCH(1),1}$ is a constant because of independent from $m$.

Figures 2 and 3 are about here.

Next we consider option pricing problems for a class of processes generated by uncorrelated random variables, which includes the linear process and an important class in time series analysis. Here we are concerned with the following process

$$X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}, \quad t \in \mathbb{Z},$$

where $\{\varepsilon_t; t \in \mathbb{Z}\}$ is a sequence of uncorrelated random variables. Instead of (A1) and (A2) we make the following assumption.

(B1) $\{\varepsilon_t; t \in \mathbb{Z}\}$ is fourth order stationary in the sense that

(i) $E(\varepsilon_t) = 0$,

(ii) $Var(\varepsilon_t) = \sigma^2$,

(iii) $\text{cum}(\varepsilon_t, \varepsilon_{t+u_1}, \varepsilon_{t+u_2}) = c_{\varepsilon,3}(u_1, u_2)$,

(iv) $\text{cum}(\varepsilon_t, \varepsilon_{t+u_1}, \varepsilon_{t+u_2}, \varepsilon_{t+u_3}) = c_{\varepsilon,4}(u_1, u_2, u_3)$.

(B2) The cumulants $c_{\varepsilon,k}(u_1, \ldots, u_{k-1})$, $k = 3, 4$, satisfy

$$\sum_{u_1, \ldots, u_{k-1} = -\infty}^{\infty} \left(1 + |u_j|^{2-k/2}\right) |c_{\varepsilon,k}(u_1, \ldots, u_{k-1})| < \infty,$$

for $j = 1, \ldots, k - 1$.

(B3) $\{a_j; j \in \mathbb{Z}\}$ satisfies

$$\sum_{j=0}^{\infty} (1 + |j|)|a_j| < \infty.$$
Under (B2), \( \{ \varepsilon_t; t \in \mathbb{Z} \} \) has the \( k \)-th order cumulant spectral density. Let \( f_{\varepsilon,k} \) be the \( k \)-th order cumulant spectral density evaluated at frequency 0

\[
f_{\varepsilon,k} = (2\pi)^{-(k-1)} \sum_{u_1, \ldots, u_{k-1} = -\infty}^{\infty} c_{\varepsilon,k}(u_1, \ldots, u_{k-1})
\]

for \( k = 2, 3, 4 \). The response function of \( \{ a_j; j \in \mathbb{Z} \} \) is defined by

\[
A(\lambda) = \sum_{j=0}^{\infty} a_j e^{-ij\lambda}
\]

for \( -\infty < \lambda < \infty \).

Under (B1)-(B3), (A1) and (A2) hold. Hence, from Theorem 1, we have

**Corollary 2.** Suppose that (B1)-(B3) and (A3) hold. Let \( a_1 = \exp(-r\tau) \) and \( a_2 = \exp \left( \tau \mu + \frac{1}{2} \tau \sigma^2 A^2 \right) \). Then

\[
C = G_0 + \frac{2\pi^2 A^3}{3\sigma^3 |A|^3} N^{-1/2} f_{\varepsilon,3} G_3 - \frac{1}{2A^2} N^{-1} f'_{\varepsilon,2} G_2 \\
+ \frac{\pi^3}{3\sigma^2} N^{-1} f_{\varepsilon,4} G_4 + \frac{2\pi^4}{9\sigma^4} N^{-1} f_{\varepsilon,5} G_6 + o(N^{-1}),
\]

where \( A = A(0) \),

\[
f'_{\varepsilon,2} = 2 \sum_{j_1, j_2 = 0}^{\infty} |j_2| a_{j_1} a_{j_1 + j_2},
\]

\( G_k, k = 0, 2, 3, 4, 6, \) are given in Theorem 2 with

\[
f_{X,2} = \frac{\sigma^2}{2\pi} A^2.
\]

**Example 4.** Let \( \{ X_t; t \in \mathbb{Z} \} \) be AR(1) process

\[
X_t = \rho X_{t-1} + \varepsilon_t, \quad |\rho| < 1.
\]

Note that

\[
A = \frac{1}{1 - \rho}, \quad f'_{\varepsilon,2} = \frac{2\rho}{(1 + \rho)(1 - \rho)^3}.
\]

The price of a European call option \( C_{AR(1)} \) is given by

\[
C_{AR(1)} = G_{AR(1),0} + N^{-1/2} G_{AR(1),2} + N^{-1} G_{AR(1),3} + o(N^{-1}),
\]
where

\[ G_{AR(1),0} = a_1 \{ a_2 S_T \Phi(d_1) - K \Phi(d_2) \}, \]
\[ G_{AR(1),2} = \frac{2\pi^2}{3\sigma_3} f_{\varepsilon,3} G_3, \]
\[ G_{AR(1),3} = -\frac{\rho}{1 - \rho^2} G_2 + \frac{\pi^3}{3\sigma^4} f_{\varepsilon,4} G_4 + \frac{2\pi^4}{9\sigma^6} (f_{\varepsilon,3})^2 G_6, \]
\[ a_2 = \exp \left\{ \tau \mu + \frac{\tau \sigma^2}{2(1 - \rho)^2} \right\}, \]
\[ d_1 = \frac{\log S_T / K + \tau \mu + \frac{\tau \sigma^2}{1 - \rho^2}}{\left( \frac{\tau^{1/2} \sigma}{1 - \rho} \right)^{1/2}}, \]
\[ d_2 = d_1 - \left( \frac{\tau^{1/2} \sigma}{1 - \rho} \right), \]

and

\[ G_k = a_1 a_2 S_T \left\{ \sum_{j=1}^{k-1} \left( \frac{\tau^{1/2} \sigma}{1 - \rho} \right)^j H_{k-j-1}(-d_2) \phi(d_1) + \left( \frac{\tau^{1/2} \sigma}{1 - \rho} \right)^k \Phi(d_1) \right\}, \]

for \( k = 2, 3, 4, 6 \).

In order to show influences of higher order terms, in Figure 4, we plotted \( C_{AR(1),0} \) (dotted line), \( C_{AR(1),2} = G_{AR(1),0} + N^{-1/2} G_{AR(1),2} \) (dashed line) and \( C_{AR(1),3} = G_{AR(1),0} + N^{-1/2} G_{AR(1),2} + N^{-1} G_{AR(1),3} \) (solid line) of Example 4 with \( S_T = K = 100, \tau = 30/365, N = 30 (\Delta = 1/365), r = \mu = 0.05, \sigma = 1, f_{X,3} = -0.1, f_{X,4} = 0.2 \) and \(-1 < \rho < 0.75\). From this, we observe that \( C_{AR(1),k}, k = 1, 2, 3 \) diverges as \( \rho \to 1 \).

Figure 4 is about here.

In Examples 2 and 3, although the third order terms diverge, the first order terms do not diverge. On the other hand, in Example 4, even the first order term does not converge as \( \rho \to 1 \). This fact is attributed to finiteness of the variances.

## 3. Martingale restriction

In the previous section, we considered pricing problems with no martingale property. Now we recall that the theoretical price of an option is based on the risk neutrality argument. In this section, to investigate influences of the martingale restriction, we derive the option price based on the risk neutrality argument (see Cox and Ross, 1976 and Longstaff, 1995).

Let
\[ d_1^* = \frac{(\log S_T / K + r \tau + \pi \tau f_{X,2})}{(2\pi \tau f_{X,2})^{1/2}}, \]
\[ d_2^* = d_1^* - (2\pi \tau f_{X,2})^{1/2}. \]
Then we have

**Theorem 3.** The fair price $C^*$ of a European call option is given by

$$C^* = G_6^* + \frac{(2\pi)^{1/2}}{6}N^{-1/2} \frac{f_{X,3}}{(f_{X,2})^{3/2}} G_3^* - \frac{1}{4\pi} N^{-1} \frac{f_{X,2} G_2^*}{f_{X,2}} + \frac{\pi}{12} N^{-1} \frac{f_{X,4}}{(f_{X,2})^2} G_4^* + \frac{\pi}{36} N^{-1} \frac{(f_{X,3})^2}{(f_{X,2})^3} G_6^* + o(N^{-1}),$$

where

$$G_0^* = S_0 \Phi(d_1^*) - e^{-r\tau} K \Phi(d_2^*),$$
$$G_k^* = S_0 \sum_{j=1}^{k-1} (2\pi\tau f_{X,2})^{j/2} H_{k-j-1}(-d_2^*) \phi(d_1^*),$$

for $k = 2, 3, 4$ and

$$G_6^* = S_0 \left[ \sum_{j=1}^{2} (2\pi\tau f_{X,2})^{j/2} \left\{ H_{5-j}(-d_2^*) - 2\pi\tau f_{X,2} H_{3-j}(-d_2^*) \right\} \right] \phi(d_1^*).$$

**Example 5.** Suppose that $\{X_t; t \in \mathbb{Z}\}$ is AR(1) process in Example 4. Then the fair price of a European call option $C_{AR(1)}^*$ is given by

$$C_{AR(1)}^* = G_{AR(1),0}^* + N^{-1/2} G_{AR(1),2}^* + N^{-1} G_{AR(1),3}^* + o(N^{-1}),$$

where

$$G_{AR(1),0}^* = S_0 \Phi(d_1^*) - e^{-r\tau} K \Phi(d_2^*),$$
$$G_{AR(1),2}^* = \frac{2\pi^3}{3\sigma^2} f_{\varepsilon,3} G_3^*,$$
$$G_{AR(1),3}^* = \frac{-\rho}{1 - \rho^2} G_2^* + \frac{\pi^3}{3\sigma^4} f_{\varepsilon,4} G_4^* + \frac{2\pi^4}{9\sigma^6} (f_{\varepsilon,3})^2 G_6^*,$$
$$d_1^* = \left\{ \log S_0 / K + \tau \tau + \frac{\tau^2}{2(1 - \rho)^2} \right\} / \left( \frac{\tau^{1/2}}{1 - \rho} \right),$$
$$d_2^* = d_1^* - \left( \frac{\tau^{1/2}}{1 - \rho} \right),$$
$$G_k^* = S_0 \left\{ \sum_{j=1}^{k-1} \left( \frac{\tau^{1/2}}{1 - \rho} \right)^j H_{k-j-1}(-d_2^*) \phi(d_1^*) \right\},$$

for $k = 2, 3, 4$ and

$$G_6^* = S_0 \left[ \sum_{j=1}^{2} \left( \frac{\tau^{1/2}}{1 - \rho} \right)^j \left\{ H_{5-j}(-d_2^*) - \frac{\tau^2}{(1 - \rho)^2} H_{3-j}(-d_2^*) \right\} \right] \phi(d_1^*).$$
In Figure 5, we plotted $C_{AR(1),1} = G_{AR(1),0}^* \ast N^{-1/2} G_{AR(1),2}^*$ (dotted line), $C_{AR(1),2} = G_{AR(1),0}^* \ast N^{-1/2} G_{AR(1),2}^*$ (dashed line) and $C_{AR(1),3} = G_{AR(1),0}^* \ast N^{-1/2} G_{AR(1),2}^* \ast N^{-1} G_{AR(1),3}^*$ (solid line) of Example 5 with $ST_0 = K = 100$, $\tau = \frac{30}{365}$, $N = 30$, $\Delta = \frac{1}{365}$, $r = 0.05$, $\sigma = 1$, $f_{X,4} = -0.1$, $f_{X,3} = 0.2$ and $-1 < \rho < 1$. Unlike Example 4, we observe that $C_{AR(1),k}, k = 1, 2, 3$ converge to $ST_0 (= 100)$ as $\rho \to 1$.

4. Estimation

From (1), $X_{j-N_0}, j = 1, \ldots, N_0$, are available, where $N_0 = T_0/\Delta$. Therefore, in this section we consider to estimate $\mu, f_{X,2}, f'_{X,2}, f_{X,3}$ and $f_{X,4}$ in Theorems 1 and 2 consistently based on the past observations. From (A1), $\Delta \mu$ is the mean of stock log returns. Hence, a natural unbiased estimator of $\mu$ is the sample mean

$$\hat{\mu} = \frac{1}{\Delta N_0} \sum_{j=1}^{N_0} \{ \log S_{j\Delta} - \log S_{(j-1)\Delta} \},$$

(10)

The variance of $\hat{\mu}$ is given by

$$Var(\hat{\mu}) = \frac{1}{\Delta N_0} \sum_{u=-(N_0-1)}^{N_0-1} \left( 1 - \frac{|u|}{N_0} \right) c_{X,2}(u).$$

Hence, under (A2), $\hat{\mu}$ given in (10) is consistent estimator of $\mu$.

Moreover in order to construct consistent estimator of $f'_{X,2}$, we define the lag window function $w(\cdot)$ which is an even and piecewise continuous function satisfying the conditions,

$$w(0) = 1,$$

$$|w(x)| \leq 1, \quad \text{for all } x,$$

$$w(x) = 0, \quad \text{for } |x| > 1.$$

(11)

Let

$$\hat{f}'_{X,2} = \sum_{u=-(N_0-1)}^{N_0-1} |u| \hat{c}_{X,2}(u) w(B_{N_0} u),$$

where $\hat{c}_{X,2}(u)$ is the sample autocovariance function at lag $u$

$$\hat{c}_{X,2}(u) = \frac{1}{\Delta N_0} \sum_{j=1}^{N_0-|u|} \{ \log S_{(j+|u|)\Delta} - \log S_{(j+|u|-1)\Delta} - \Delta \hat{\mu} \} \times \{ \log S_{j\Delta} - \log S_{(j-1)\Delta} - \Delta \hat{\mu} \},$$

(12)

and $B_{N_0} \to 0$ as $N_0 \to \infty$, but $(B_{N_0})^3 N_0 \to \infty$. Then we can easily see that under (A2), $\hat{f}'_{X,2}$ given in (12) is a consistent estimator of $f'_{X,2}$. 11
Since $f_{X,k}$, $k = 2, 3, 4$, are the $k$-th order cumulant spectral density evaluated at frequency 0, using Brillinger and Rosenblatt (1967a and 1967b) formula, we construct consistent estimators $\hat{f}_{X,k}$ of $f_{X,k}$ ($k = 2, 3, 4$). See also Brillinger (1981). Thus we can consistently estimate all the quantities in Theorems 1 and 2 (e.g., $G_j$, $j = 0, 2, 3, 4, 6$) by the corresponding quantities replacing $\mu$, $f'_{X,2}$ and $f_{X,k}$ by $\hat{\mu}$, $\hat{f}'_{X,2}$ and $\hat{f}_{X,k}$ ($k = 2, 3, 4$).

For example, we discuss a consistent estimator for New York stock exchange data. The data are daily returns of AMOCO, FORD HP, IBM and MERCK companies. The individual time series are the last 1024 data points from stocks, representing the daily returns for the five companies from February 2, 1984, to December 31, 1991. We used the window functions

$$W(u_1, \ldots, u_{k-1}) = \begin{cases} 2^{-(k-1)} & \text{if } |u_1|, \ldots, |u_{k-1}| \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

for $\hat{f}_{X,k}$ ($k = 2, 3, 4$) and Let $w(u) = 1$ for $|u| \leq 1$, where $w(u)$ is defined in (11). Also we used the bandwidth in frequency direction with $B_{N_0} = 1/50$ for $\hat{f}_{X,2}$, $B_{N_0} = 1/30$ for $\hat{f}_{X,3}$ and $B_{N_0} = 1/10$ for $\hat{f}_{X,4}$ and $\hat{f}'_{X,2}$ (see Brillinger and Rosenblatt 1967a and 1967b, and Brillinger 1981).

Table 1: Values of Consistent estimators

<table>
<thead>
<tr>
<th></th>
<th>AMOCO</th>
<th>FORD</th>
<th>HP</th>
<th>IBM</th>
<th>MERCK</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\mu}$</td>
<td>0.235103</td>
<td>0.045337</td>
<td>0.133815</td>
<td>0.017165</td>
<td>0.481340</td>
</tr>
<tr>
<td>$f_{X,2}$</td>
<td>0.002937</td>
<td>0.016006</td>
<td>0.016202</td>
<td>0.003085</td>
<td>0.004534</td>
</tr>
<tr>
<td>$f_{X,3}$</td>
<td>-0.706149</td>
<td>-3.078889</td>
<td>8.501363</td>
<td>0.470144</td>
<td>2.419969</td>
</tr>
<tr>
<td>$f_{X,4}$</td>
<td>2.278478</td>
<td>-0.280973</td>
<td>8.651378</td>
<td>15.0914</td>
<td>-2.249174</td>
</tr>
<tr>
<td>$\frac{f_{X,2}}{f_{X,2}}$</td>
<td>-22.78799</td>
<td>-5.520428</td>
<td>0.169291</td>
<td>27.18047</td>
<td>-37.3221</td>
</tr>
</tbody>
</table>

Table 1 show these values of consistent estimators of $\mu$, $f'_{X,2}$ and $f_{X,k}$ ($k = 2, 3, 4$) for the five companies. From this results, we can see that the quantities involved in higher order terms is quite different from the Black and Scholes model. Therefore, in general the assumptions of the Gaussianity and the independence of stock log returns will not hold.

Table 2: Option prices

<table>
<thead>
<tr>
<th></th>
<th>AMOCO</th>
<th>FORD</th>
<th>HP</th>
<th>IBM</th>
<th>MERCK</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>2.776419</td>
<td>4.031663</td>
<td>4.472833</td>
<td>1.699889</td>
<td>4.689151</td>
</tr>
<tr>
<td>$C_2$</td>
<td>2.809884</td>
<td>3.979554</td>
<td>4.434833</td>
<td>1.700269</td>
<td>4.495491</td>
</tr>
<tr>
<td>$C_3$</td>
<td>2.881406</td>
<td>4.345765</td>
<td>6.392765</td>
<td>1.374588</td>
<td>4.650024</td>
</tr>
</tbody>
</table>
Table 2 show these values of the approximation up to the first $C_1$, second $C_2$ and third order $C_3$ of the option prices with $S_{T_0} = K = 100$, $\tau = 30/365$, $N = 30$, $r = 0.05$. From this results, we observe that option prices are strongly affected by third order terms except for AMOCO and MERCK.

Table 3: Fair prices

<table>
<thead>
<tr>
<th></th>
<th>AMOCO</th>
<th>FORD</th>
<th>HP</th>
<th>IBM</th>
<th>MERCK</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1^*$</td>
<td>1.764254</td>
<td>3.827175</td>
<td>3.849221</td>
<td>1.699889</td>
<td>2.138307</td>
</tr>
<tr>
<td>$C_2^*$</td>
<td>1.769475</td>
<td>3.784867</td>
<td>3.954549</td>
<td>1.700269</td>
<td>2.124842</td>
</tr>
<tr>
<td>$C_3^*$</td>
<td>1.83751</td>
<td>4.111153</td>
<td>6.09142</td>
<td>1.374588</td>
<td>2.459177</td>
</tr>
</tbody>
</table>

Table 3 show these values of the approximation up to the first $C_1^*$, second $C_2^*$ and third order $C_3^*$ of the fair prices with $S_{T_0} = K = 100$, $\tau = 30/365$, $N = 30$, $r = 0.05$. From this results, we observe that option prices are strongly affected by third order terms.

5. Concluding remark

The Black and Scholes model assumes the Gaussianity and the independency of stock log returns. Empirical studies, however, report that they are not Gaussian nor independent. In this paper, dropping these two assumptions, we derive a European option pricing. Then, we observed that option prices are strongly affected by the non-Gaussianity and the dependency of stock log returns. Hence, it should be noted that we use option pricing models taking account of the non-Gaussianity and the dependency of stock log returns.

References


Figure 1: For t-distribution with \( \nu \) degrees of freedom in Example 2, the approximation up to the first \( (C_{t,1}, \text{dotted line}) \) and third order \( (C_{t,3}, \text{solid line}) \) of the option price are plotted with \( S_{T_0} = K = 100, \tau = 30/365, N = 30, \) \( r = \mu = 0.05 \) and \( 4 < \nu < 9. \)
Figure 2: For ARCH(1) in Example 3, the approximation up to the first ($C_{ARCH(1),1}$, dotted line) and third order ($C_{ARCH(1),3}$, solid line) of the option price are plotted with $S_{T_0} = K = 100$, $\tau = 30/365$, $N = 30$, $r = \mu = 0.05$, $m = 3$, $\psi_0 = 0.5$ and $-1/\sqrt{3} < \psi_1 < 1/\sqrt{3}$.
Figure 3: For ARCH(1) in Example 3, the approximation up to the first $(C_{ARCH(1)}, 1$, dotted line) and third order $(C_{ARCH(1)}, 3$, solid line) of the option price are plotted with $S_{T_0} = 100$, $K = 95$, $\tau = 30/365$, $N = 30$, $r = \mu = 0.05$, $\psi_0 = 0.5$, $\psi_1 = 0.3$ and $1 < m < 9$. 
Figure 4: For AR(1) in Example 4, the approximation up to the first (\(C_{ARCH(1),1}\), dotted line), second (\(C_{ARCH(1),2}\), dashed line) and third order (\(C_{ARCH(1),3}\), solid line) of the option price are plotted with \(S_0 = K = 100\), \(\tau = 30/365\), \(N = 30\), \(r = \mu = 0.05\), \(\sigma = 1\), \(f_{X,3} = -0.1\), \(f_{X,4} = 0.2\) and \(-1 < \rho < 0.75\).
Figure 5: For AR(1) in Example 5, the approximation up to the first \( (C^{*}_{ARCH(1),1}, \text{dotted line}) \), second \( (C^{*}_{ARCH(1),2}, \text{dashed line}) \) and third order \( (C^{*}_{ARCH(1),3}, \text{solid line}) \) of the option price are plotted with \( S_{T_0} = K = 100 \), \( \tau = 30/365 \), \( N = 30 \), \( r = 0.05 \), \( \sigma = 1 \), \( f_{X,3} = -0.1 \), \( f_{X,4} = 0.2 \) and \(-1 < \rho < 1\).