The cusum test in time series models

Sangyeol Lee
Department of Statistics
Seoul National University
Seoul, 151-742, Korea
email: sylee@stats.snu.ac.kr

ABSTRACT

In this paper we consider the problem of testing for a scale change in the infinite order moving average process, for the parameters in unstable models, and for those of regression models with ARCH errors. For performing the test, the cusum of squares test analogous to Inclán and Tiao (1994)’s statistic is considered.

1. Introduction

Since economic time series are frequently affected by events such as changes in fiscal or monetary policy, the problem of testing the parameter constancy of a time series has received considerable attention from researchers. Inclán and Tiao (1994) proposed the cusum of squares test for testing a variance change in iid normal r.v.’s, based on the earlier work by Brown, Durbin and Evans (1975) that deals with the problem of testing the constancy of the regression coefficients in regression models. In Inclán and Tiao, provided the observations $\varepsilon_t$ are given, the variance change is detected based on the test statistic:

\[
IT_n = \max_{1 \leq k \leq n} \left( \frac{n}{2} \right)^{1/2} \left| \frac{\sum_{j=1}^{k} \varepsilon_j^2}{\sum_{j=1}^{n} \varepsilon_j^2} - \frac{k}{n} \right|.
\]

A large value of $IT_n$ indicates the existence of a variance change. The critical values are obtainable asymptotically, since $IT_n$ has the same limiting distribution as $\sup_{0 \leq t \leq 1} |B^n(t)|$, where $B^n$ is a standard Brownian bridge. In this talk, we discuss on testing an abrupt change of the variance in MA($\infty$) (infinite order moving average) processes, unstable models with unit roots, and regression models with ARCH errors based on the cusum test. For the references, we refer to Lee and Park (2002), Lee, Na and Na (2003), and Lee, Tokutsu and Maekawa (2004), and the cited papers therein.
2. Cusum of squares test

In this section we introduce the cusum of squares test and give the related asymptotic results. Let \( \{X_j\} \) be the MA(\( \infty \)) process such that

\[
X_j = \sum_{i=0}^{\infty} a_i \varepsilon_{j-i},
\]

where \( \varepsilon_i \) are iid r.v.’s. In obtaining the limiting distribution of the test statistic, Donsker’s invariance principle for dependent processes plays an important role as we mentioned earlier. Here, we do not assume any mixing conditions for \( \{X_j\} \) since it is well-known that \( \{X_j\} \) is not necessarily strongly mixing. Instead, we adopt the mixingale approach introduced by McLeish.

First, we extend Inclán and Tiao’s test to the process in (2.1). To this end, we assume that \( E\varepsilon_1 = 0, E\varepsilon_1^2 = \sigma^2, E\varepsilon_1^8 < \infty \) and \( \sum_{i=0}^{\infty} |a_i| < \infty \). Suppose that \( X_1, \cdots, X_n \) are observed and let \( \nu = EX_1^2, \gamma(h) = E(X_1^2 - \nu)(X_{1+h}^2 - \nu) \) for \( h = 0, \pm 1, \cdots \), and \( \varphi^2 = \sum_{h=-\infty}^{\infty} \gamma(h) \). Define \( \hat{\nu} = n^{-1} \sum_{i=1}^{n} X_i^2 \) and

\[
\hat{\gamma}(h) = n^{-1} \sum_{i=1}^{n-|h|} (X_i^2 - \hat{\nu})(X_{i+h}^2 - \hat{\nu}) \quad \text{for } |h| < n.
\]

The following theorem shows that the test statistic \( T_n \) in (2.3) below, constructed in analogy to \( IT_n \) in (1.1), has the same limiting distribution as \( IT_n \) under certain regularity conditions.

**Theorem 2.1.** Suppose that \( E|\varepsilon_1|^8 < \infty \) and \( |a_i| \leq ci^{-q} \) for some \( c > 0 \) and \( q > 5/2 \). Assume that a sequence of positive integers \( \{h_n\} \) satisfies

\[
H : \ h_n \to \infty \quad \text{and} \quad h_n = O(n^\rho) \quad \text{for some } \rho \in (0, 1/2).
\]

Then, as \( n \to \infty \),

\[
\varphi^2 := \sum_{|h| \leq h_n} \hat{\gamma}(h) \xrightarrow{P} \varphi^2,
\]

and

\[
T_n := \max_{1 \leq k \leq n} \frac{n^{1/2}\hat{\nu}}{\varphi} \left| \frac{\sum_{i=1}^{k} X_i^2}{\sum_{i=1}^{n} X_i^2} - \frac{k}{n} \right| \xrightarrow{d} \sup_{0 \leq t \leq 1} \left| B^\alpha(t) \right|,
\]

where \( B^\alpha \) denotes a standard Brownian bridge.

In the remainder of this section, we develop the cusum of squares test based on trimmed observations in order to handle the case where the data is contaminated by outliers such as gross
errors, or has the characteristics of a marginal distribution with infinite variance. Towards this end, we consider the model to accommodate outliers as follows:

\[(2.5) \quad U_j = (1 - p_j)X_j + p_j V_j,\]

where \(p_j\) are iid r.v.'s with \(0 \leq p_j \leq 1\), the contaminating process \(\{V_j\}\) is a sequence of iid r.v.'s with \(EV_j = 0\) and \(EV_j^2 < \infty\), and \(\{p_j\}\), \(\{V_j\}\) and \(\{X_j\}\) are all independent. In this case, unlike before, we do not apply the eighth moment condition to \(\varepsilon_1\). Instead, we assume that 

\[E|\varepsilon_1|^\alpha < \infty \quad \text{for some} \quad \alpha > 0.\]

Also, we assume that \(\sum_{i=0}^{\infty} |a_i|^\alpha < \infty\) and \(\sum_{i=0}^{\infty} |a_i| < \infty\) according to whether \(\alpha < 1\) or \(\alpha \geq 1\). Here, we consider the case of \(\alpha < 2\) to accommodate infinite variance processes.

Note that (2.5) becomes an I.O. (innovation outlier) model when \(p_j = 0\) for all \(j\) and the errors \(\varepsilon_j\) follow a heavy-tailed non-Gaussian distribution, and an A.O. (additive outlier) model when \(\alpha \geq 2\) and \(p_j = 1/2\) for all \(j\) and \(V_j\) has an appropriate distribution. It also indicates a S.O. (substitutive outliers) model when \(\alpha \geq 2\) and \(p_j\) are Bernoulli r.v.'s.

Let \(F\) denote the distribution of \(U_1\) and assume that the density \(f = F'\) satisfies

\[(2.6) \quad R : f(x) > 0 \quad \text{for all} \quad x \quad \text{and} \quad \sup_x f(x) < \infty.\]

For \(u \in (0, 1)\), let \(\xi_u\) be a number such that \(F(\xi_u) = u\). Provided \(U_1, \ldots, U_n\) are given, set

\[(2.7) \quad \xi_{nu} = \begin{cases} U_{(n, \lfloor nu \rfloor)}, & \text{\(nu\) is an integer} \\ U_{(n, \lfloor nu \rfloor + 1)}, & \text{\(nu\) is not an integer} \end{cases},\]

where \(U_{(n1)}, \ldots, U_{(nn)}\) denote the ordered r.v.'s of \(U_1, \ldots, U_n\), and \([x]\) is the largest integer not exceeding \(x\). Let \(u < v\) be numbers in \((0, 1)\). We denote \(\Psi_j^2 = U_j^2 I(\xi_{nu} \leq U_j \leq \xi_{nv})\), \(\mu^* = n^{-1} \sum_{j=1}^{n} \Psi_j^2\), and \((\tau^*)^2 = \sum_{|h| \leq h_n} \gamma^*(h)\), where

\[\gamma^*(h) = n^{-1} \sum_{i=1}^{n-|h|} (\Psi_{i}^2 - \mu^*)(\Psi_{i+|h|}^2 - \mu^*) \quad \text{for} \quad |h| < n,\]

and \(\{h_n\}\) is a sequence of positive integers satisfying (2.2).

**Theorem 2.2.** Suppose that \(E|\varepsilon_1|^\alpha < \infty\), \(|a_i| \leq c_i^{-q}\) for some \(\alpha, q > 0\) with \((\alpha \land 1)q > 7\), and \(h_n\) satisfies (2.2) with \(\rho \in (0, 3/8)\). Let

\[T_n^* = \frac{n^{1/2} \mu^*}{\tau^*} \max_{1 \leq k \leq n} \left| \frac{\sum_{j=1}^{k} \Psi_j^2}{\sum_{j=1}^{n} \Psi_j^2} - \frac{k}{n} \right| .\]

Then, under Condition \(R\),

\[(2.8) \quad T_n^* \d \sup_{0 \leq t \leq 1} |B^\alpha(t)| \quad \text{as} \quad n \to \infty.\]
3. Test for AR($q$) models

In this section we consider the problem of testing for a variance change in the unstable AR($q$) model:

\[ X_t - \beta_1 X_{t-1} - \cdots - \beta_q X_{t-q} = \epsilon_t, \]  

where $\epsilon_t$ are iid random variables with $E\epsilon_1 = 0$, $E\epsilon_1^2 = \sigma^2$ and $E\epsilon_1^4 < \infty$. We assume that the corresponding characteristic polynomial $\phi$ has a decomposition
\[
\phi(z) = 1 - \beta_1 z - \cdots - \beta_q z^q = (1 - z)^a (1 + z)^b \prod_{k=1}^{l} (1 - 2 \cos \theta_k z + z^2)^{d_k} \psi(z),
\]

where $a, b, l, d_k$ are nonnegative integers, $\theta_k$ belongs to $(0, \pi)$ and $\psi(z)$ is the polynomial of order $r = q - (a + b + 2d_1 + \ldots + 2d_l)$ that has no zeros on the unit disk in the complex plane.

Let $X_t = (X_t, \ldots, X_{t-q+1})'$, where $X_t = 0$ for all $t \leq 0$. Let

\[
\hat{\beta}_n = \left( \sum_{t=1}^{n} X_{t-1}X_{t-1}' \right)^{-1} \sum_{t=1}^{n} X_{t-1}X_t, \quad n > q,
\]

be the least squares estimate of $\beta = (\beta_1, \ldots, \beta_q)'$ based on $X_1, \ldots, X_n$. Then the residuals are

\[
\hat{\epsilon}_t = X_t - \hat{\beta}_n X_{t-1}, \quad t = 1, \ldots, n.
\]

As mentioned earlier, our goal is to test the following hypotheses:

$H_0$ : the $\epsilon_t$ have the same variance $\sigma^2$ vs.

$H_1$ : not $H_0$.

In order to perform a test, we employ the cusum of squares test statistic $T_n$ based on the residuals:

\[ T_n = \frac{1}{\sqrt{n\kappa_n}} \max_{1 \leq k \leq n} \left| \sum_{t=1}^{k} \hat{\epsilon}_t^2 - \frac{k}{n} \sum_{t=1}^{n} \hat{\epsilon}_t^2 \right|, \tag{3.2} \]

where $\kappa_n^2 = n^{-1} \sum_{t=1}^{n} \hat{\epsilon}_t^4 - (n^{-1} \sum_{t=1}^{n} \hat{\epsilon}_t^2)^2$. Then we have the following result.

**Theorem 0.1** Under $H_0$, as $n \to \infty$,

\[ T_n \overset{w}{\to} \sup_{0 \leq u \leq 1} |W^\circ(u)|, \tag{3.3} \]

where $W^\circ$ denotes a standard Brownian bridge. We reject $H_0$ if $T_n$ is large.
4. Test for regression models with ARCH errors

Let us consider the model

\[ y_t = \beta' z_t + \epsilon_t, \]

(4.1)

\[ \epsilon_t = h_t \xi_t, \]

\[ h_t^2 = a(\theta) + \sum_{j=1}^{\infty} b_j(\theta) \epsilon_{t-j}^2, \]

where \( \xi_t \) are iid r.v.’s with zero mean and unit variance, \( \{z_t\} \) is a \( p \)-dimensional strictly stationary process, and \( \theta \rightarrow a(\theta) \) and \( \theta \rightarrow b(\theta) \) are nonnegative continuous real functions defined on a subset \( \mathcal{N} \) in \( \mathbb{R}^d \) with \( a(\theta) > 0 \) and \( \sum_{j=1}^{\infty} b_j(\theta) < \infty \) for all \( \theta \in \mathcal{N} \). We assume that \( y_s, z_s, s < t \) are independent of \( \xi_u, u \geq t \), and \( \{(\epsilon_t, h_t, z_t)\} \) is strong mixing. The Model (1) covers a broad class of important models in the financial time series context including GARCH models. In particular, it becomes a GARCH(1,1) model if we put \( z_t = 0, \theta = (\omega, \alpha_1, \alpha_2), \omega, \alpha_1, \alpha_2 > 0, \alpha_1 + \alpha_2 < 1, a(\theta) = \omega/(1 - \alpha_2) \) and \( b_j(\theta) = \alpha_1 \alpha_j^{1-j} \). In this case, \( \{(\epsilon_t, h_t, z_t)\} \) is geometrically strong mixing.

The objective here is to test the hypotheses

\[ H_0 : \eta = (\beta', \theta')' \text{ remains the same for the whole series } \quad \text{vs.} \]

\[ H_1 : \text{Not } H_0. \]

For a test, one may construct a cusum test based on \( \{\hat{\epsilon}_t := y_t - \hat{\beta}' z_t\} \) as in Inclán and Tiao (1994) and Kim, Cho and Lee (2000). However, as observed in the simulation study in Section 3, the test in GARCH(1,1) models is unstable and produces low powers. Thus one has to develop a better test which is not much affected by the GARCH parameters. As a candidate, one can naturally consider the cusum test based on \( \{\xi_t^2\} \), say,

\[ T_n := \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \left| \frac{k}{n} \sum_{t=1}^{k} \xi_t^2 - \left( \frac{k}{n} \right) \sum_{t=1}^{n} \xi_t^2 \right|, \]

(4.2)

where \( \tau^2 = Var(\xi_1^2) \), since \( T_n \) is free from the GARCH parameters. In this case, however, one may speculate whether \( T_n \) can detect any changes since \( T_n \) itself has no information about the GARCH parameters. But since \( \xi_t \) are not observable, one should replace \( \xi_t^2 \)'s by the residuals \( \hat{\xi}_t^2 \), which are obtained via estimating the unknown parameters. Those estimators play an important role to detect changes in the parameters in the presence of changes, while the iid property of the true errors still remains when there are no changes. From this reasoning, one can anticipate that the residual cusum test should be more stable and produce better powers.
Now, we construct the residual cusum test. To this end, we assume that

(A1) $E|z_1|^{4+\delta_1} < \infty$, $E|\xi_1|^{4+\delta_1} < \infty$ and $E|\xi_1|^{4+\delta_1} < \infty$ for some $\delta_1 > 0$.

(A2) There exists $\delta_2 > 0$ such that

$$\sup_{||\theta - \theta'|| \leq \delta_2, \theta' \in \mathcal{N}} ||\hat{a}(\theta)|| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \sup_{||\theta - \theta'|| \leq \delta_2, \theta' \in \mathcal{N}} ||\hat{b}_j(\theta)|| < \infty,$$

where $\hat{a}(\theta)$ and $\hat{b}_j(\theta)$ denote the gradient vectors of $a$ and $b_j$ at $\theta$.

(A3) There exists a sequence of positive integers with $q \to \infty$, $q/\sqrt{n} \to 0$ and

$$\sqrt{n} \sum_{j=q+1}^{\infty} b_j(\theta) \to 0 \quad \text{as} \quad n \to \infty.$$

(A4) $\{(\epsilon_t, h_t, z_t)\}$ is strong mixing with order $\gamma(h) \geq 1$ satisfying $\sum_{h=1}^{\infty} \gamma(h) \frac{\delta_1}{4+\delta_1} < \infty$.

Observe that the last condition in (A3) is satisfied if $b_j(\theta)$ are geometrically bounded (as in GARCH models), and $q = \lfloor \log n \rfloor^c$, $c > 1$. Also, if $z_t$ are identically zero and $\{y_t\}$ is a GARCH process, $\{(y_t, h_t)\}$ is geometrically strong mixing, so that (A4) is satisfied.

Now, we construct the residual cusum test. In analogy of $h_t^2$, we define

$$h_t^2 = a(\hat{\theta}) + \sum_{j=1}^{q} b_j(\hat{\theta})\hat{\epsilon}_{t-j}^2,$$

$$\hat{\epsilon}_t = y_t - \hat{\theta}' z_t \quad \text{and} \quad \hat{\xi}_t = \hat{\epsilon}_t/\hat{h}_t;$$

where $\hat{\eta} = (\hat{\beta}', \hat{\theta}')$ is an estimator of $\eta$ with $\sqrt{n}(\hat{\eta} - \eta) = O_P(1)$. Then, we have the following result.

**Theorem 0.2** Assume that (A1)-(A4) hold. Set

$$\hat{T}_n := \frac{1}{\sqrt{n \hat{\sigma}^2}} \max_{q+1 \leq k \leq n} \left| \sum_{t=q+1}^{k} \hat{\xi}_t^2 - \left( \frac{k}{n} \right) \sum_{t=q+1}^{n} \hat{\xi}_t^2 \right|$$

where $\hat{\sigma}^2 = \frac{1}{n-q} \sum_{t=q+1}^{n} \hat{\xi}_t^2 - \left( \frac{1}{n-q} \sum_{t=q+1}^{n} \hat{\xi}_t^2 \right)^2$. Then, under $H_0$,

$$\hat{T}_n \overset{d}{\to} \sup_{0 \leq u \leq 1} |B^u(u)|, \quad n \to \infty,$$

where $B^u$ is a Brownian bridge.

**REFERENCES**

