Local Whittle Likelihood Estimators and Tests for non-Gaussian Linear Processes

Abbreviated Title : Local Whittle Likelihhod for Linear Processes

By Tomohito NAITO, Kohei ASAI, Tomoyuki AMANO and Masanobu TANIGUCHI*

Department of Mathematical Sciences,
School of Science and Engineering, Waseda University

Abstract

In this paper, we propose a local Whittle likelihood estimator for spectral densities of non-Gaussian linear processes and a local Whittle likelihood ratio test statistic for the problem of testing whether the spectral density of a non-Gaussian stationary linear process belongs to a parametric family or not. Introducing a local Whittle likelihood of a spectral density $f_\theta(\lambda)$ around λ, we propose a local estimator $\hat{\theta} = \hat{\theta}(\lambda)$ of θ which maximizes the local Whittle likelihood around λ, and use $f_{\hat{\theta}(\lambda)}(\lambda)$ as an estimator of the true spectral density. For the testing problem, we use a local Whittle likelihood ratio test statistic based on the local Whittle likelihood estimator. The asymptotics of these statistics are elucidated. It is shown that their asymotototic distributions do not depend on non-Gaussianity of the processes. Because our models include nonlinear stationary time series models, we can apply the results to stationary GARCH processes. Advantage of the proposed estimator is demonstrated by a few simulated numerical examples.

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*Corresponding author.

1 Introduction

The study of spectral density functions of stationary processes provides an effective approach to estimate their underlying models and various methods for testing problems. There are two ways to estimate the spectral density directly. The one is the parametric method which is studied in the literature (e.g., Brockwell and Davis (1991)). Also Hosoya and Taniguchi (1982) constructed a very general framework of the Whittle estimator for a class of non-Gaussian linear processes. Further, Giraitis and Robinson (2001) proposed to use the Whittle estimation procedure for the squared ARCH processes. The other is the nonparametric method based on smoothed periodogram (e.g., Hannan (1970)).

For i.i.d. observations, Hjort and Jones (1996) proposed a new probability density estimator $f_{\hat{\theta}(x)}(x)$ which maximizes a local likelihood for f_θ around x. They showed that $f_{\hat{\theta}(x)}(x)$ has the same asymptotic variance as the ordinary nonparametric kernel estimator but potentially a smaller bias. For a Gaussian stationary process, Fan and Kreutzberger (1998) proposed a local polynomial estimator based on the Whittle likelihood. Then it was shown that it has advantages over the least-squares based on log-periodogram.

In this paper, for a class of non-Gaussian linear processes, we introduce a local Whittle likelihood of the spectral density $f_\theta(\lambda)$ and propose the local Whittle estimator $f_{\hat{\theta}(\lambda)}(\lambda)$ around each frequency $\lambda \in [-\pi, \pi]$. Then we elucidate the asymptotics of $\hat{\theta}(\lambda)$ and $f_{\hat{\theta}(\lambda)}(\lambda)$.

Next we consider the problem of testing whether the spectral density of a class of stationary processes belongs to a parametric family or not. For this testing problem, Fan and Yao (2003, Section 9.3.2) and Fan and Zhang (2004) proposed a generalized likelihood ratio tests based on the Whittle likelihood and a local Whittle estimator. Then they elucidated the asymptotics of the generalized likelihood ratio tests under the null hypothesis.

Because the results above rely on Gaussianity of the process concerned, in this paper, we drop this assumption, and discuss the problem of testing whether the spectral density of a class of non-Gaussian linear processes belongs to a parametric family or not. A local Whittle likelihood ratio test is proposed. Then it is shown that the asymptotic distribution of the test converges in distribution to a normal distribution.

An interesting feature is that the asymptotics of the estimator and test statistic do not depend on non-Gaussianity of the process. Because we do not assume Gaussianity of the process concerned, it is possible to apply the results to stationary nonlinear processes which include GARCH processes. Numerical studies for the local Whittle likelihood estimator are provided.

It may be noted that Robinson (1995) discussed a semiparametric estimation of long memory processes, and his approach is a local Whittle likelihood estimation for long memory parameters around the origin i.e., $\lambda = 0$.

This paper is organized as follows. Section 2 describes the model, assumptions and estimator $\hat{\theta}(\lambda)$. In Section 3, we derive the asymptotics of $\hat{\theta}(\lambda)$ and $f_{\hat{\theta}(\lambda)}(\lambda)$. Section 4 introduces test statistic, then the asymptotic distribution of it is derived under the null hypothesis. Numerical examples for a class of MA(1) processes are provided in Section 5. They illuminate some interesting features of our approach. The proofs of the main theorems are relegated to Section 6. Throughout this paper, we denote the set of all integers by J, and denote Kronecker's delta by $\delta(m, n)$.

2 Setting

In this section we describe the model, assumptions and estimators. Throughout this paper we consider a class of non-Gaussian linear processes, which include not only ARMA but also squared GARCH processes etc .

Assume that $\{z(n) : n \in J\}$ is a general linear process defined by

$$z(n) = \sum_{j=0}^{\infty} G(j)e(n-j), \quad n \in J, \tag{2.1}$$

where $\{e(n)\}$ is a white noise process satisfying

$$E[e(n)] = 0$$
$$E[e(n)e(m)] = \delta(m, n)\sigma^2, \quad E[e(n)^4] < \infty.$$

Furtheremore, we assume that

$$\sum_{j=0}^{\infty} G(j)^2 < \infty. \tag{2.2}$$

Then $\{z(n)\}$ is a second-order stationary process with spectral density

$$f(w) = \frac{\sigma^2}{2\pi} \left| \sum_{j=0}^{\infty} G(j)e^{-iwj} \right|^2 . \tag{2.3}$$

Henceforth we denote by $\mathcal{P}$ the set of all spectral density funcitons of the form (2.3). Write the autocovariance function of $z(n)$ as $\gamma(\cdot)$. Then we assume,

Assumption 1.

$$\sum_{-\infty}^{\infty} n^2|\gamma(n)| < \infty. \tag{2.4}$$

Let $z(1), z(2), \ldots, z(N)$ be an observed stretch of $\{z(n)\}$, and denote the periodogram of $\{z(n)\}$ by

$$I_N(\lambda) = \frac{1}{2\pi N} \left| \sum_{n=1}^{N} z(n)e^{in\lambda} \right|^2 .$$

Assumption 2. Let $K(\cdot)$ be a kernel function which satisfies:

(1) K is a real bounded nonnegative even function.

(2) $\int_{-\infty}^{\infty} K(t)dt = 1, \quad \int_{-\infty}^{\infty} t^2 K(t)dt = \sigma_k^2 < \infty, \quad \int_{-\infty}^{\infty} s^2 K^2(s)ds < \infty$

The condition $\int_{-\infty}^{\infty} t^2 K(t)dt < \infty$ implies

$$\lim_{x\to 0} \frac{1 - k(x)}{x^2} < \infty \tag{2.5}$$

where $k(x) = \int_{-\infty}^{\infty} K(t)e^{itx}dt$. Here (2.5) leads to (4.7) in Hannan (1970, p.283) with his $q = 2$.

Let $\{f_\theta(\lambda) : \theta \in \Theta \subset R^q\}$ be a parametric family of spectral density functions of $\mathcal{P}$ where Θ is a compact set. Here we impose the following assumption.

Assumption 3. $f_\theta(w)$ is three times continuously differentiable with respect to w and θ, and there exists $\delta > 0$ such that $f_\theta(\omega) \geq \delta$ for all $\omega \in [-\pi, \pi]$.

Assumption 4. For $\theta_1,\ \theta_2 \in \Theta$, if $\theta_1 \neq \theta_2$, then $f_{\theta_1} \neq f_{\theta_2}$ on a set of Lebesgue measure.

We define a local distance function $D_\lambda(\cdot, \cdot)$ around a given local point λ for spectral densities $\{f_t\}$ by

$$D_\lambda(f_t, f) = \int_{-\pi}^{\pi} K_h(\lambda - w) \left\{ \log f_t(w) + \frac{f(w)}{f_t(w)} \right\} dw,$$

where $K_h(x) = \frac{1}{h}K(x/h)$, and $h > 0$ is a bandwidth. If we replace f by I_N, we call $D_\lambda(f_\theta, I_N)$ the local Whittle likelihood function under f_θ. Here the misspecification of f_θ for f is allowed.

Define $T_{\lambda,h}(f) \in \Theta$ by

$$D_\lambda(f_{T_{\lambda,h}(f)}, f) = \min_{t\in\Theta} D_\lambda(f_t, f),$$

Henceforth we sometimes write $T_{\lambda,h}(f)$ as $\theta_h(\lambda)$ which is called a pseudo-true value of θ. As an estimator of $\theta_h(\lambda)$, we use $\hat{\theta}_h(\lambda)$ defined by

$$\hat{\theta}_h(\lambda) = T_{\lambda,h}(I_N) = \arg\min_{t\in\Theta} D_\lambda(f_t, I_N).$$

We can use $f_{\hat{\theta}_h(\lambda)}(\lambda)$ as a local estimator of f and call this the local Whittle likelihood estimator. For simplicity we sometimes write $f_{\theta_h(\lambda)}(\lambda)$ as $f_{\theta_h}(\lambda)$.

Remark 2.1. Robinson (1995) developed semiparametric estimation of long memory processes with

$$f(\omega) \sim G\omega^{1-2H}, \quad as \quad \omega \to 0+, \tag{2.6}$$

where H and G are unknown parameters. Although our model does not include (2.6), his approach corresponds to the case when

$$K(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases} \text{ (Daniell window)}$$

and $\lambda = 0$ in $D_\lambda(f_t, I_N)$.

3 Estimating Theory

In this section, we investigate the asymptotics of $\hat{\theta}_h(\lambda)$ and $f_{\hat{\theta}_h}(\lambda)$. Fan and Kreutsberger (1998) showed the asymptotics of a local polynomial estimator of the spectral density based on the Whittle likelihood for Gaussian linear processes. Here we set down the following assumption.

Assumption 5. $\sum_{j_1,j_2,j_3=-\infty}^{\infty} |Q_e(j_1, j_2, j_3)| < \infty$ where $Q_e(j_1, j_2, j_3)$ is the joint fourth order cumulant of $e(n), e(n + j_1), e(n + j_2), e(n + j_3)$.

Next we show the asymptotic distribition of $\hat{\theta}_h(\lambda)$ as $N \to \infty$.

Theorem 3.1. *Suppose that the* $\{z(n)\}$ *given in (2.1) and* $K(\cdot)$ *satisfy Assumptions 1-5,* $T_{\lambda,h}(g)$ *exists uniquely and lies in Int* Θ*, and that*

$$M_h(\lambda) = \int_{-\pi}^{\pi} K_h(\lambda - t)\frac{\partial^2}{\partial\theta\partial\theta'}(f_{\theta_h}^{-1}(\omega)f(\omega) + \log f_{\theta_h}(\omega))d\omega$$

is a nonsingular matrix for every $h > 0$*. Then if* $N \to \infty$*,*

$$\sqrt{N}\Big\{\hat{\theta}_h(\lambda) - \theta_h(\lambda)\Big\} \longrightarrow N(0, M_h(\lambda)^{-1}\tilde{V}_h(M_h(\lambda)^{-1})^{\tau})$$

where

$$\begin{aligned}\tilde{V}_h &= 2\pi\int_{-\pi}^{\pi} K_h(\lambda-\omega)^2\frac{\partial}{\partial\theta}f_{\theta_h}^{-1}(\omega)\frac{\partial}{\partial\theta'}f_{\theta_h}^{-1}(\omega)f^2(\omega)d\omega \\ &+ 2\pi\int_{-\pi}^{\pi} K_h(\lambda-\omega)K_h(\lambda+\omega)\frac{\partial}{\partial\theta}f_{\theta_h}^{-1}(\omega)\frac{\partial}{\partial\theta'}f_{\theta_h}^{-1}(\omega)f^2(\omega)d\omega \\ &+ 2\pi\int\int_{-\pi}^{\pi} K_h(\lambda-\omega_1)K_h(\lambda+\omega_2)\frac{\partial}{\partial\theta}f_{\theta_h}^{-1}(\omega_1)\frac{\partial}{\partial\theta'}f_{\theta_h}^{-1}(\omega_2)Q^z(\omega_1,\omega_2,-\omega_2)d\omega_1 d\omega_2.\end{aligned} \tag{3.1}$$

Here $Q^z(,,)$ *is the fourth-order cummualnt spectral density of* $\{z(t)\}$.

Until now, we have assumed that $h > 0$ is fixed. In what follows we assume that h is a function of N satisfying

Assumption 6.

$$(N^{\frac{1}{2}}h)^{-1} + N^{\frac{1}{5}}h \longrightarrow 0 \quad as\ N \to \infty.$$

The following theorem establishes the asymptotic normality of local Whittle estimator $f_{\hat{\theta}_h(\lambda)}(\lambda)$.

Theorem 3.2. *Under Assumptions 1-6,*

$$\sqrt{Nh}(f_{\hat{\theta}_h(\lambda)}(\lambda) - f(\lambda)) \xrightarrow{d} N(0, \Psi),$$

where

$$\Psi = \begin{cases} 2\pi \int_{-\infty}^{\infty} K^2(s)ds f_{\theta_0}(\lambda)^2, & (\lambda \neq 0) \\ 4\pi \int_{-\infty}^{\infty} K^2(s)ds f_{\theta_0}(\lambda)^2 & (\lambda = 0). \end{cases}$$

The proofs of the theorems are relegated to Section 5.

Remark 3.1 In Theorem 3.2, we can see that the asymptotic variance and bias of the local Whittle estimator depend on only $f(\lambda)$, $f_{\theta_0}(\lambda)$ and K. Thus the asymptotic distribution of the local Whittle estimator does not depend on non-Gaussianity of the process.

4 Testing Theory

When we estimate the spectral density of an observed time series, it is a significant problem whether the spectral density is parametric or not. For this, Fan and Zhang (2004) applies local linear polynomial technique to the log-periodogram of Gaussian process.

Consider the problem of testing whether $f(\lambda)$ belongs to a specific parametric family $\{f_\theta(\cdot) : \theta \in \Theta\}$ or not, i.e.,

$$H_0 : f(\cdot) = f_\theta(\cdot) \quad \text{v.s.} \quad H_1 : f(\cdot) \neq f_\theta(\cdot). \tag{4.1}$$

Although we do not assume Gaussianity of $\{z_t\}$, the Whittle likelihood function under H_0 is expressed as

$$l(\theta) = -\sum_{k=1}^{N} \left\{ \log f_\theta(\lambda_k) + \frac{I_N(\lambda_k)}{f_\theta(\lambda_k)} \right\},\ \lambda_k = -\pi + 2\pi k/N\ (k = 1, \cdots, N).$$

We call $\hat{\theta}_{\mathrm{WH}} = \arg\max_{\theta\in\Theta} l(\theta)$ the Whittle likelihood estimator of θ.

In this paper, for H_1, we use the following local Whittle likelihood function around $\lambda \in [-\pi, \pi]$:

$$l^{\mathrm{loc}}(\theta) = -\sum_{k=1}^{N} \left\{ \log f_\theta(\lambda_k) + \frac{I_N(\lambda_k)}{f_\theta(\lambda_k)} \right\} K_h(\lambda - \lambda_k), \tag{4.2}$$

where $K_h(\cdot)$ is an appropriate kernel function. Let $\hat{\theta}_{\mathrm{LW}}(\lambda) = \arg\max_{\theta\in\Theta} l^{\mathrm{loc}}(\theta)$, and we regard $f_{\hat{\theta}_{\mathrm{LW}}(\lambda)}(\lambda)$ as a sort of nonparametric estimator of the spectral density $f(\lambda)$. For the testing problem (4.1), we use the following likelihood ratio test statistic

$$\begin{aligned} T_{\mathrm{LW}} &= l(\hat{\theta}_{\mathrm{WH}}) - l(\hat{\theta}_{\mathrm{LW}}(\lambda)) \\ &= -\sum_{k=1}^{N} \left\{ \log f_{\hat{\theta}_{\mathrm{WH}}}(\lambda_k) + \frac{I_N(\lambda_k)}{f_{\hat{\theta}_{\mathrm{WH}}}(\lambda_k)} \right\} + \sum_{k=1}^{N} \left\{ \log f_{\hat{\theta}_{\mathrm{LW}}(\lambda_k)}(\lambda_k) + \frac{I_N(\lambda_k)}{f_{\hat{\theta}_{\mathrm{LW}}(\lambda_k)}(\lambda_k)} \right\} \\ &= \sum_{k=1}^{N} \left\{ \log f_{\hat{\theta}_{\mathrm{LW}}(\lambda_k)}(\lambda_k) - \log f_{\hat{\theta}_{\mathrm{WH}}}(\lambda_k) + I_N(\lambda_k)(f_{\hat{\theta}_{\mathrm{LW}}(\lambda_k)}(\lambda_k)^{-1} - f_{\hat{\theta}_{\mathrm{WH}}}(\lambda_k)^{-1}) \right\}. \end{aligned}$$

Actually if $T_{\mathrm{LW}} > z_\alpha$, a selected level, we reject H_0, otherwise, accept it.

We derive the asymptotics of T_{LW} under H_0 of (4.1). Write T_{LW} as

$$\begin{aligned} T_{\mathrm{LW}} &= \{l(\theta) - l(\hat{\theta}_{\mathrm{LW}}(\lambda))\} - \{l(\theta) - l(\hat{\theta}_{\mathrm{WH}})\} \\ &= T_{\mathrm{LW},1} - T_{\mathrm{LW},2}. \end{aligned}$$

It is known that $T_{\mathrm{LW},2} = O_P(1)$ under appropriate regularity conditions (e.g., Taniguchi and Kakizawa (2000, Section 3.1)). In what follows, it is seen that $T_{\mathrm{LW},1}$ is asymptotically of order in probability tending to ∞. Hence, in order to derive the asymptotic distribution of T_{LW} we have only to derive that of $T_{\mathrm{LW},1}$. For this, furthermore, we impose the following assumption.

Assumption 7. (i) For each k, $k = 2, 3, \cdots$, $\{z(t)\}$ is kth-order stationary with all of whose moments exist.
(ii) The joint kth-order cumulant $Q_z(j_1, \cdots, j_{k-1})$ of $z(t), z(t+j_1), \cdots, z(t+j_{k-1})$ satisfies

$$\sum_{j_1,\cdots,j_{k-1}=-\infty}^{\infty} (1 + |j_l|)|Q_z(j_1, \cdots, j_{k-1})| < \infty$$

for $l = 1, \cdots, k-1$ and any k, $k = 2, 3, \cdots$.

Then we get the following theorem whose proof is relegated to Section 5

Theorem 4.1. *Suppose that Assumptions 1-4 and 6-7 hold. Then, under H_0,*

$$\sigma_N^{-1}(T_{\mathrm{LW}} - \mu_N) \to^d N(0,1),$$

where

$$\mu_N = \frac{1}{h}\left[-2\pi K(0) + \pi \int_{-\infty}^{\infty} K(\omega)^2 d\omega\right]$$
$$\sigma_N^2 = \frac{4\pi}{h}\int_{-\infty}^{\infty} K(\omega)^2 d\omega.$$

Remark 4.1. *It should be noted that the asymptotics of T_{LW} also do not depend on non-Gaussianity of the process and spectra. This seems interesting.*

5 Numerical Examples

In this section we give the numerical study of the local Whittle likelihood estimator. In the simulation we compare the MSE of the local Whittle likelihood estimators and smoothed periodogram estimators with some kernel functions. The MSE of the spectral estimator by use of the maximum likelihood estimator is also compared. Then it is seen that the local Whittle likelihood estimators are better than the other estimators.

Consider the following MA(1) model:

$$z(t) = \epsilon(t) + 0.2\epsilon(t-1) \tag{5.1}$$

where $\epsilon(t) \sim i.i.d.N(0,1)$ and its spectral density $f(\lambda)$ is given by

$$f(\lambda) = \frac{1}{2\pi}\left|1 + 0.2e^{i\lambda}\right|^2. \tag{5.2}$$

To estimate $f(\lambda)$, we consider the local Whittle likelihood estimator by fitting a family of spectral densities $\{f_\theta, \theta \in \Theta\}$ given by

$$f_\theta(\lambda) = \frac{1}{2\pi}|1 - \theta e^{i\lambda}|^{-2} = \frac{1}{2\pi}(1 - 2\theta\cos\lambda + \theta^2)^{-1}. \tag{5.3}$$

The integral of local Whittle likelihood is approximated by the sum of Fourier frequencies $w_n = 2\pi n/N$ $(1 \le n \le N)$. For $I_N(\lambda) = \frac{1}{2\pi N}|\sum_{n=1}^N z(n)e^{in\lambda}|^2$, we calculate $D_\lambda(f_\theta, I_N)$ over grid points on $(-1,1)$ for θ to derive $\hat{\theta}(\lambda)$ for each λ. We compare performances of the local Whittle estimators with the smoothed periodogram estimators,

$$\hat{f}(\lambda) = \int_{-\pi}^{\pi} K_h(\lambda - t) I_N(t) dt. \tag{5.4}$$

By fitting AR(30) model to (5.1) the spectral estimator $f_{\hat{\theta}_{MLE}}(\lambda)$ is also compared where $\hat{\theta}_{MLE}$ is the maximum likelihood estimator for autoregressive coefficients. In Table 1 below, based on 5000 times simulations and $N = 1000$, we provide the values of MSE of the local Whittle likelihood estimators and the smoothed periodogram estimators with kernel functions with $h = 0.3$ (Daniell, Tukey-Hanning, Parzen, Abel) (see Hannan (1970), p278-p279).

	$\lambda = \frac{1}{5}\pi$	$\lambda = \frac{2}{5}\pi$	$\lambda = \frac{3}{5}\pi$	$\lambda = \frac{4}{5}\pi$	$\lambda = \pi$
$f_{\hat{\theta}_h(\lambda)}(\lambda)$(Daniell)	0.000289	0.00021	0.000124	0.000088	0.000148
$\hat{f}(\lambda)$(Daniell)	0.00031	0.000234	0.000148	0.000098	0.002447
$f_{\hat{\theta}_h(\lambda)}(\lambda)$(Tukey-Hanning)	0.000222	0.000163	0.000106	0.000066	0.000098
$\hat{f}(\lambda)$(Tukey-Hanning)	0.000299	0.000110	0.000056	0.000150	0.002019
$f_{\hat{\theta}_h(\lambda)}(\lambda)$(Parzen)	0.000195	0.000159	0.000086	0.000049	0.000098
$\hat{f}(\lambda)$(Parzen)	0.000549	0.000184	0.000058	0.000154	0.001514
$f_{\hat{\theta}_h(\lambda)}(\lambda)$(Abel)	0.000231	0.000155	0.00008	0.000071	0.000107
$\hat{f}(\lambda)$(Abel)	0.000726	0.000387	0.000153	0.000117	0.002028
$f_{\hat{\theta}_{MLE}}(\lambda)$	0.003121	0.002298	0.001421	0.000837	0.001525

Table 1: MSE of local Whittle likelihood estimators, smoothed periodogram estimators, the spectral estimator by fitting AR(m)

From this table we observe that the local Whittle likelihood estimators are better than the other estimators.

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Tomohito, NAITO.
Department of Mathematical Sciences
School of Science and Engineering
Waseda University
3-4-1 Okubo Shinjuku-ku
Tokyo 169-8555
Japan.
E-mail: tomohito-n@toki.waseda.jp

Kohei, ASAI.
School of Science and Engineering
3-4-1 Okubo Shinjuku-ku
Japan.
E-mail: asai@toki.waseda.jp

Tomoyuki, AMANO.
Department of Applied Mathematics
School of Fundamental Science and Engineering
Waseda University
3-4-1 Okubo Shinjuku-ku

Tokyo 169-8555
Japan.
E-mail: tomtochami@aoni.waseda.jp

Masanobu, TANIGUCHI.
Department of Mathematical Sciences
School of Science and Engineering
Waseda University
3-4-1 Okubo Shinjuku-ku
Tokyo 169-8555
Japan.
E-mail: taniguchi@waseda.jp