# Generalized Cordeiro–Ferrari Bartlett-type adjustment

Yoshihide KAKIZAWA (Hokkaido University)

**Abstract**

The Bartlett-type adjustment is a higher-order asymptotic method for improving the chi-squared approximation to the null distributions of various test statistics, which ensures that the resulting test has size $\alpha + \mathrm{o}(N^{-1})$, where $0 < \alpha < 1$ is the significance level and $N$ is the sample size. Three influential papers were published in 1991; Chandra and Mukerjee (CM), Cordeiro and Ferrari (CF) and Taniguchi (T) in alphabetical order. Recently, author re-investigated the CM/T-approaches in a unified way and then derived the $N^{-1}$-local powers of several Bartlett-type adjustments* in the presence of nuisance parameters. This paper considers a generalization of the CF-approach.

## 1. Introduction

It is well known that the null distributions of the likelihood ratio (LR), Rao's and Wald's test statistics have asymptotic expansions in powers of $N^{-1}$, where $N$ is the sample size. The Bartlett (or Bartlett-type) adjustment is designed to make the chi-squared approximation accurate up to order $N^{-1}$. Historically, for the LR test statistic $\mathrm{LR}^{(N)}$, the following fact was first exploited by Bartlett (1937) in his classical test for homogeneity of variances: "A simple mean adjustment for $\mathrm{LR}^{(N)}$ through multiplication by a constant of the form $1 + b/N$ implies $P^{(N)}[(1 + b/N)\mathrm{LR}^{(N)} \leq x] = \Pr[\chi_f^2 \leq x] + \mathrm{o}(N^{-1})$ under the null hypothesis", where $f$ is the number of restrictions under test. After Lawley (1956), this became widely known as the Bartlett correctability of $\mathrm{LR}^{(N)}$. Among the vast literature, we further mention Hayakawa (1977), Bickel and Ghosh (1990), Jensen (1993) and Kakizawa (2011) for the theoretical issues.

Cox (1988) argued that Rao's and Wald's test statistics are generally not Bartlett correctable (see also Taniguchi (1988) and Bickel and Ghosh (1990)). Barndorff-Nielsen and Cox (1994; page 132) thus posed a question whether there is an effective general way of improving the approximations to the null distributions of the test statistics other than the LR test statistic. Chandra and Mukerjee (1991) and Taniguchi (1991b) proposed Bartlett-type adjustments for the test statistic $T^{(N)}$ (they assumed that $T^{(N)}$ admits the $N^{-1}$-stochastic expansion), on the basis of the information of score or maximum likelihood estimator (MLE), respectively. Cordeiro and Ferrari (1991) gave a polynomial transformed test statistic** $T^{\mathrm{CF}(N)} = \{1 + N^{-1}\sum_{\ell=1}^{k} c_\ell (T^{(N)})^{\ell-1}\}T^{(N)}$ (see also Kakizawa (1996) for the corresponding monotone version), where $k \in \mathbf{N}$ and the coefficients $c_\ell$'s are determined according to an asymptotic expansion for the null distribution of $T^{(N)}$. Even in the case where the $c_\ell$'s involve unknown parameters, they might be replaced by suitable estimators without affecting the order of the approximation. Note that for the case $k = 1$, $T^{\mathrm{CF}(N)}$ is the traditional Bartlett adjustment. However, the several attempts in Chandra and Mukerjee (1991) and Taniguchi (1991b) (see also Kakizawa (2010)) were restricted to the test of the simple null hypothesis. Thus, extending Mukerjee (1992), who considered Rao's test statistic in the presence of a scalar nuisance parameter, Kakizawa (2012b) re-examined the Bartlett-type adjustments for a class of test statistics in the framework of a composite hypothesis about a subvector of the parameters*, where both the parameter of interest and the nuisance parameter are multidimensional, without the assumption of the global parameter orthogonality.

A contribution of this paper (Section 3) is to generalize the Cordeiro-Ferrari adjustment in a sense. Another contribution of this paper (Section 4) is to study the $N^{-1}$-local power after the generalized Cordeiro-Ferrari adjustment. As a result, we show that in general, the $N^{-1}$-local power after the original

*In Kyoto symposium (November 30, 2011), author discussed the $N^{-1}$-local power properties after the Bartlett-type adjustments proposed in Kakizawa (2012b), using the methodology from the CM/T-approaches.

**The literature on the Cordeiro-Ferrari adjustment with $k = 2, 3$ is very extensive; some related papers during the last two decades are found in Kakizawa (2012b).

Cordeiro-Ferrari adjustment is the same as the $N^{-1}$-local power of the size adjusted test (based on the Cornish-Fisher expansion of the percentile). This statement is consistent with Magdalinos (1994) on several tests for the admissibility of a subset of instrumental variables and Kakizawa (2009) on the normal-based GMANOVA tests under a non-Gaussian error.

## 2. Preliminaries

Although we focus on an iid model for notational simplicity, we will arrive at the same conclusions even in a non-identical or dependent model where some regularity conditions are met for the log-likelihood derivatives according to the situations under consideration.

### 2.1. Notation

Let $\mathbf{X}_1, \ldots, \mathbf{X}_N$ be iid random vectors (taking values of $\mathbf{R}^{d_X}$) according to a density $f(\mathbf{x}, \boldsymbol{\theta})$, $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, where $\boldsymbol{\Theta}$ is an open convex subset of $\mathbf{R}^p$. We assume that the parameter $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)'$ is composed of two parts, a parameter of interest $\boldsymbol{\theta}_{(1)} = (\theta_1, \ldots, \theta_{p_1})'$ and a nuisance parameter $\boldsymbol{\theta}_{(2)} = (\theta_{p_1+1}, \ldots, \theta_{p_1+p_2})'$; $\boldsymbol{\theta} = (\boldsymbol{\theta}'_{(1)}, \boldsymbol{\theta}'_{(2)})' \in \boldsymbol{\Theta} = \boldsymbol{\Theta}_{(1)} \times \boldsymbol{\Theta}_{(2)}$ (say), where $p = p_1 + p_2$. We write $\mathcal{L}^{(N)}(\boldsymbol{\theta}) = \sum_{i=1}^N \log f(\mathbf{X}_i, \boldsymbol{\theta})$. We want to test a composite hypothesis $\boldsymbol{\theta}_{(1)} = \boldsymbol{\theta}_{(1)0}$ against $\boldsymbol{\theta}_{(1)} \neq \boldsymbol{\theta}_{(1)0}$, where $\boldsymbol{\theta}_{(1)0} \in \boldsymbol{\Theta}_{(1)}$ is specified while $\boldsymbol{\theta}_{(2)} \in \boldsymbol{\Theta}_{(2)}$ remains unspecified. Let $\widehat{\boldsymbol{\theta}}^{(N)}_{\mathrm{ML}} \in \boldsymbol{\Theta}$ be the (unrestricted) MLE of $\boldsymbol{\theta}$, and let $\widetilde{\boldsymbol{\theta}}^{(N)}_{(2)\mathrm{ML}} \in \boldsymbol{\Theta}_{(2)}$ be the restricted MLE of $\boldsymbol{\theta}_{(2)}$ under the constraint $\boldsymbol{\theta}_{(1)} = \boldsymbol{\theta}_{(1)0}$, where we write $\widetilde{\boldsymbol{\theta}}^{(N)}_{\mathrm{ML}} = \begin{pmatrix} \boldsymbol{\theta}_{(1)0} \\ \widetilde{\boldsymbol{\theta}}^{(N)}_{(2)\mathrm{ML}} \end{pmatrix}$.

As usual, the $R$th partial derivative of the log density $\log f(\mathbf{x}, \boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ is denoted by

$$\ell_{j_1 \ldots j_R}(\mathbf{x}, \boldsymbol{\theta}) = \frac{\partial}{\partial \theta_{j_1}} \cdots \frac{\partial}{\partial \theta_{j_R}} \log f(\mathbf{x}, \boldsymbol{\theta}) \quad \text{for } R \in \mathbf{N};\ j_1, \ldots, j_R \in \{1, \ldots, p\}.$$

We introduce $I_R = j_1 \ldots j_R$ for notational simplicity and denote the cumulants of the $\ell_{I_R}(\mathbf{X}, \boldsymbol{\theta})$'s by

$$\nu_{I_{R_1}, \ldots, I_{R_v}}(\boldsymbol{\theta}) = Cum_\theta[\ell_{I_{R_1}}(\mathbf{X}, \boldsymbol{\theta}), \ldots, \ell_{I_{R_v}}(\mathbf{X}, \boldsymbol{\theta})]$$

(descending order $R_1 \geq \cdots \geq R_v \geq 1$ on the size $R_i = |I_{R_i}|$ is assumed, since $\nu_{I_{R_1}, \ldots, I_{R_v}}(\boldsymbol{\theta})$ is symmetric under permutation of $\{I_{R_1}, \ldots, I_{R_v}\}$). We assume that the Bartlett identities hold, i.e.

$$\nu_{j_1}(\boldsymbol{\theta}) = 0, \quad \nu_{j_1 j_2}(\boldsymbol{\theta}) + \nu_{j_1, j_2}(\boldsymbol{\theta}) = 0, \quad \nu_{j_1 j_2 j_3}(\boldsymbol{\theta}) + \langle 3 \rangle \nu_{j_1 j_2, j_3}(\boldsymbol{\theta}) + \nu_{j_1, j_2, j_3}(\boldsymbol{\theta}) = 0,$$
$$\nu_{j_1 j_2 j_3 j_4}(\boldsymbol{\theta}) + \langle 4 \rangle \nu_{j_1 j_2 j_3, j_4}(\boldsymbol{\theta}) + \langle 3 \rangle \nu_{j_1 j_2, j_3 j_4}(\boldsymbol{\theta}) + \langle 6 \rangle \nu_{j_1 j_2, j_3, j_4}(\boldsymbol{\theta}) + \nu_{j_1, j_2, j_3, j_4}(\boldsymbol{\theta}) = 0$$

($\langle n \rangle$ before a term with indices is a sum of $n$ similar terms obtained by index permutation) for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. According to the partition $\boldsymbol{\theta} = (\boldsymbol{\theta}'_{(1)}, \boldsymbol{\theta}'_{(2)})'$, we stack $\nu_{j,k}(\boldsymbol{\theta})$ and $Z_j^{(N)}(\boldsymbol{\theta}) = N^{-1/2} \sum_{i=1}^N \ell_j(\mathbf{X}_i, \boldsymbol{\theta})$ as follows: $[\nu_{j,k}(\boldsymbol{\theta})]_{j,k \in \{1,\ldots,p\}} = \begin{pmatrix} \boldsymbol{\nu}_{(1,1)}(\boldsymbol{\theta}) & \boldsymbol{\nu}_{(1,2)}(\boldsymbol{\theta}) \\ \boldsymbol{\nu}_{(2,1)}(\boldsymbol{\theta}) & \boldsymbol{\nu}_{(2,2)}(\boldsymbol{\theta}) \end{pmatrix}$, $[Z_j^{(N)}(\boldsymbol{\theta})]_{j=1,\ldots,p} = \begin{pmatrix} \mathbf{Z}^{(N)}_{(1)}(\boldsymbol{\theta}) \\ \mathbf{Z}^{(N)}_{(2)}(\boldsymbol{\theta}) \end{pmatrix}$ (they are referred to as the $p \times p$ Fisher information matrix $\boldsymbol{\nu}(\boldsymbol{\theta})$ and the $p \times 1$ score vector $\mathbf{Z}^{(N)}(\boldsymbol{\theta})$, respectively). Further, we write $Z^{(N)}_{j_1 \ldots j_R}(\boldsymbol{\theta}) = N^{-1/2} \sum_{i=1}^N \{\ell_{j_1 \ldots j_R}(\mathbf{X}_i, \boldsymbol{\theta}) - \nu_{j_1 \ldots j_R}(\boldsymbol{\theta})\}$, $R = 2, 3, \ldots$.

Unless otherwise stated, we use the letters $\{j, k\}$ as indices of $\boldsymbol{\theta}$ that run from 1 to $p$, the letters $\{a, b\}$ as indices of $\boldsymbol{\theta}_{(1)}$ that run from 1 to $p_1$ and the letters $\{r, s\}$ as indices of $\boldsymbol{\theta}_{(2)}$ that run from $p_1 + 1$ to $p$. These indices, without or with suffixes (or primes), serve two purposes, first to denote a typical element of any $R$-way array $[Q_{j_1 \ldots j_R}]_{j_1, \ldots, j_R \in \{1, \ldots, p\}}$ and second to indicate the range of a sum in the Einstein summation convention. We denote by $\nu^{j,k}(\boldsymbol{\theta})$ the $(j, k)$th element of $\boldsymbol{\nu}^{-1}(\boldsymbol{\theta})$, where we assume that $\boldsymbol{\nu}(\boldsymbol{\theta})$ is nonsingular in this paper. Let $[\nu_{(2,2)}^{r,s}(\boldsymbol{\theta})]_{r,s\in\{p_1+1,\ldots,p\}}$ be the inverse of the matrix $\boldsymbol{\nu}_{(2,2)}(\boldsymbol{\theta})=[\nu_{r,s}(\boldsymbol{\theta})]_{r,s\in\{p_1+1,\ldots,p\}}$. We denote by $\nu_{(11\cdot 2)}^{a,b}(\boldsymbol{\theta})$ the $(a,b)$th element of $\boldsymbol{\nu}_{(11\cdot 2)}^{-1}(\boldsymbol{\theta})$, where

$$\boldsymbol{\nu}_{(11\cdot 2)}(\boldsymbol{\theta})=[\nu_{(11\cdot 2)a,a'}(\boldsymbol{\theta})]_{a,a'\in\{1,\ldots,p_1\}}=\boldsymbol{\nu}_{(1,1)}(\boldsymbol{\theta})-\boldsymbol{\nu}_{(1,2)}(\boldsymbol{\theta})\boldsymbol{\nu}_{(2,2)}^{-1}(\boldsymbol{\theta})\boldsymbol{\nu}_{(2,1)}(\boldsymbol{\theta})\,.$$

Further, we denote by $\mathcal{G}_{j,a}(\boldsymbol{\theta})$ the $(j,a)$th element of $\boldsymbol{\mathcal{G}}(\boldsymbol{\theta})=\begin{pmatrix}\mathbf{I}_{p_1}\\ -\boldsymbol{\nu}_{(2,2)}^{-1}(\boldsymbol{\theta})\boldsymbol{\nu}_{(2,1)}(\boldsymbol{\theta})\end{pmatrix}$, where $\mathbf{I}_{p_1}$ is the $p_1\times p_1$ identity matrix. We note $\boldsymbol{\nu}_{(11\cdot 2)}(\boldsymbol{\theta})=\boldsymbol{\mathcal{G}}(\boldsymbol{\theta})'\boldsymbol{\nu}(\boldsymbol{\theta})\boldsymbol{\mathcal{G}}(\boldsymbol{\theta})$ and

$$\boldsymbol{\mathcal{G}}(\boldsymbol{\theta})\boldsymbol{\nu}_{(11\cdot 2)}^{-1}(\boldsymbol{\theta})\boldsymbol{\mathcal{G}}(\boldsymbol{\theta})'=\boldsymbol{\nu}^{-1}(\boldsymbol{\theta})-\begin{pmatrix}\mathbf{O}_{p_1p_1} & \mathbf{O}_{p_1p_2}\\ \mathbf{O}_{p_2p_1} & \boldsymbol{\nu}_{(2,2)}^{-1}(\boldsymbol{\theta})\end{pmatrix}=[B_{j,j'}(\boldsymbol{\theta})]_{j,j'\in\{1,\ldots,p\}}\quad\text{(say)}. \tag{1}$$

*2.2. A class of (unadjusted) test statistics*

We denote by $P_\theta^{(N)}$ the $\boldsymbol{\theta}$-distribution of $\mathbf{X}_1,\ldots,\mathbf{X}_N$. For any sequence $\{Y^{(N)}\}_{N\geq 1}$ of random variables having the form $Y^{(N)}=g_N(\mathbf{X}_1,\ldots,\mathbf{X}_N)$, we use the pointwise notation $Y^{(N)}=\mathrm{o}_\theta^{(N)}(q,\beta)$ under $P_\theta^{(N)}$, if $P_\theta^{(N)}[|Y^{(N)}|>d(\log N)^\beta]=\mathrm{o}(N^{-q})$ as $N\to\infty$ for some $d>0$, $q\geq 0$ and $\beta\geq 0$.

A crucial point of the $N^{-1}$-asymptotic theory in this paper is that using Bhattacharya and Ghosh's (1978) argument and Taniguchi (1991a; page 76), $\widehat{\boldsymbol{\theta}}_{\mathrm{ML}}^{(N)}$ (and $\widetilde{\boldsymbol{\theta}}_{\mathrm{ML}}^{(N)}$) is well-defined as the local maximum point (lying in an $\epsilon^{(N)}$-neighborhood of the true parameter $\boldsymbol{\theta}^\dagger\in\boldsymbol{\Theta}$; $\epsilon^{(N)}\propto N^{-1/2}(\log N)^{1/2}$) of the unrestricted (and restricted) log-likelihood with $P_{\theta^\dagger}^{(N)}$-probability $1-\mathrm{o}(N^{-(1+\delta/2)})$ for some $\delta\geq 0^\ddagger$. For any (nonrandom/random) scalar or vector or matrix function $Q(\cdot)$, we use the notation $\widehat{Q}$, $\widetilde{Q}$ and $Q$ instead of $Q(\widehat{\boldsymbol{\theta}}_{\mathrm{ML}}^{(N)})$, $Q(\widetilde{\boldsymbol{\theta}}_{\mathrm{ML}}^{(N)})$ and $Q(\boldsymbol{\theta}^\dagger)$, respectively, where $\boldsymbol{\theta}^\dagger=\begin{pmatrix}\boldsymbol{\theta}_{(1)0}\\ \boldsymbol{\theta}_{(2)}^\dagger\end{pmatrix}\in\boldsymbol{\Theta}$, with $\boldsymbol{\theta}_{(2)}^\dagger$ being the irrelevant true value of the nuisance parameter $\boldsymbol{\theta}_{(2)}$.

Following Kakizawa (2012b), we consider a class $\mathcal{T}_N$ of test statistics for testing the null hypothesis $\boldsymbol{\theta}_{(1)}=\boldsymbol{\theta}_{(1)0}$ against $\boldsymbol{\theta}_{(1)}\neq\boldsymbol{\theta}_{(1)0}$, as follows: Every test statistic $T^{(N)}=T_N(\mathbf{X}_1,\ldots,\mathbf{X}_N;\boldsymbol{\theta}_{(1)0})\in\mathcal{T}_N$ admits a stochastic expansion of the form

$$T^{(N)}=\widetilde{T}_{\mathrm{3rd}}^{(N)}+\frac{1}{N^{3/2}}\,\mathrm{o}_{\theta^\dagger}^{(N)}(1+\xi,\beta)\quad\text{for some fixed }\beta>0\text{ and }\xi\geq 0^\ddagger \tag{2}$$

(we emphasize that $\widetilde{T}_{\mathrm{3rd}}^{(N)}=T_{\mathrm{3rd}}^{(N)}(\widetilde{\boldsymbol{\theta}}_{\mathrm{ML}}^{(N)})$ is also a feasible statistic), where

$$\begin{aligned}\widetilde{T}_{\mathrm{3rd}}^{(N)}&=(\widetilde{\mathbf{Z}}_{(1)}^{(N)})'\widetilde{\boldsymbol{\nu}}_{(11\cdot 2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)}+\frac{2}{N^{1/2}}\Big(\widetilde{C}_{b_1b_2b_3}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}\prod_{i=1}^{3}[\widetilde{\boldsymbol{\nu}}_{(11\cdot 2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)}]_{b_i}+\widetilde{C}_{b_1b_2,k_1k_2}^{\mathcal{G}\,\mathcal{G}}\widetilde{Z}_{k_1k_2}^{(N)}\prod_{i=1}^{2}[\widetilde{\boldsymbol{\nu}}_{(11\cdot 2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)}]_{b_i}\Big)\\&\quad+\frac{2}{N}\Big\{\widetilde{D}_{b_1b_2b_3b_4}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}\prod_{i=1}^{4}[\widetilde{\boldsymbol{\nu}}_{(11\cdot 2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)}]_{b_i}+(\widetilde{D}_{b_1b_2b_3,k_1k_2}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}\widetilde{Z}_{k_1k_2}^{(N)}+\widetilde{D}_{b_1b_2b_3,k_1k_2k_3}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}\widetilde{Z}_{k_1k_2k_3}^{(N)})\prod_{i=1}^{3}[\widetilde{\boldsymbol{\nu}}_{(11\cdot 2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)}]_{b_i}\\&\quad+\widetilde{D}_{b_1b_2,k_1k_2,k_3k_4}^{\mathcal{G}\,\mathcal{G}}\widetilde{Z}_{k_1k_2}^{(N)}\widetilde{Z}_{k_3k_4}^{(N)}\prod_{i=1}^{2}[\widetilde{\boldsymbol{\nu}}_{(11\cdot 2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)}]_{b_i}\Big\}\end{aligned} \tag{3}$$

(hereafter, $[\mathbf{v}]_i$ sometimes stands for the $i$th element $v_i$ of any vector $\mathbf{v}$),

$$C_{a_1a_2a_3}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}(\cdot)=C_{j_1j_2j_3}(\cdot)\mathcal{G}_{j_1,a_1}(\cdot)\mathcal{G}_{j_2,a_2}(\cdot)\mathcal{G}_{j_3,a_3}(\cdot)\,,\quad C_{a_1a_2,k_1k_2}^{\mathcal{G}\,\mathcal{G}}(\cdot)=C_{j_1j_2,k_1k_2}(\cdot)\mathcal{G}_{j_1,a_1}(\cdot)\mathcal{G}_{j_2,a_2}(\cdot)$$

(we adopt similar definitions for $D_{a_1a_2a_3a_4}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}(\cdot)$, $D_{a_1a_2a_3,k_1k_2}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}(\cdot)$, $D_{a_1a_2a_3,k_1k_2k_3}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}(\cdot)$ and $D_{a_1a_2,k_1k_2,k_3k_4}^{\mathcal{G}\,\mathcal{G}}(\cdot)$). Here, $\beta$, $\xi$, $C$-functions; $C_{j_1j_2j_3}(\cdot)$ and $C_{j_1j_2,k_1k_2}(\cdot)$, $D$-functions; $D_{j_1j_2j_3j_4}(\cdot)$, $D_{j_1j_2j_3,k_1k_2}(\cdot)$, $D_{j_1j_2j_3,k_1k_2k_3}(\cdot)$ and

‡Technically, we need $\delta,\xi>0$ for the power analysis (Section 4) under a sequence of local alternatives $\theta=\theta^\dagger+N^{-1/2}\mathrm{h}$. Assuming the usual regularity conditions, we obtained the stochastic expansions of $\widehat{\theta}_{\mathrm{ML}}^{(N)}$ and $\widetilde{\theta}_{\mathrm{ML}}^{(N)}$, as well as the connection between $\widehat{\theta}_{\mathrm{ML}}^{(N)}$ and $\widetilde{\theta}_{\mathrm{ML}}^{(N)}$. The details are found in Kakizawa (2012b).

$D_{j_1j_2,k_1k_2,k_3k_4}(\cdot)$ may vary from one test statistic in $\mathcal{T}_N$ to another, where these $C$(or $D$)-functions are of class $\mathcal{C}^2(\boldsymbol{\Theta})$ (or $\mathcal{C}^1(\boldsymbol{\Theta})$). Without loss of generality, we assume that $C_{j_1j_2j_3}(\cdot)$, $C_{j_1j_2,k_1k_2}(\cdot)$, $D_{j_1j_2j_3j_4}(\cdot)$, $D_{j_1j_2j_3,k_1k_2}(\cdot)$, $D_{j_1j_2j_3,k_1k_2k_3}(\cdot)$ and $D_{j_1j_2,k_1k_2,k_3k_4}(\cdot)$ are symmetric under permutation of $\{j_1,j_2,j_3,j_4\}$ and that $D_{j_1j_2,k_1k_2,k_3k_4}(\cdot)=D_{j_1j_2,k_3k_4,k_1k_2}(\cdot)$. Notice that this class $\mathcal{T}_N$ includes, in particular, the LR, Rao's and Wald's test statistics (the respective $C,D$-functions are found in Kakizawa (2012b)), as well as Terrell's gradient test statistic defined by $\mathrm{grad}^{(N)}=(\widetilde{\mathbf{Z}}^{(N)}_{(1)})'N^{1/2}(\widehat{\boldsymbol{\theta}}^{(N)}_{(1)\mathrm{ML}}-\boldsymbol{\theta}_{(1)0})$ with $\widehat{\boldsymbol{\theta}}^{(N)}_{\mathrm{ML}}=\begin{pmatrix}\widehat{\boldsymbol{\theta}}^{(N)}_{(1)\mathrm{ML}}\\ \widehat{\boldsymbol{\theta}}^{(N)}_{(2)\mathrm{ML}}\end{pmatrix}$.

### *2.3. Cordeiro-Ferrari adjustment*

We write

$$\begin{aligned}
C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3}(\cdot)&=\frac{1}{3!}\sum_{\{a_1a_2a_3\}}\{C^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3}(\cdot)+C^{\mathcal{G}\,\mathcal{G}}_{a_1a_2,k_1k_2}(\cdot)\nu_{k_1k_2,a_3}^{\phantom{k_1k_2,}\mathcal{G}}(\cdot)\}\\
&=C^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3}(\cdot)+\frac{\langle 3\rangle}{3}\,C^{\mathcal{G}\,\mathcal{G}}_{a_1a_2,k_1k_2}(\cdot)\nu_{k_1k_2,a_3}^{\phantom{k_1k_2,}\mathcal{G}}(\cdot)\,,\\
D^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3a_4}(\cdot)&=\frac{1}{4!}\sum_{\{a_1a_2a_3a_4\}}\{D^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3a_4}(\cdot)+D^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3,k_1k_2}(\cdot)\nu_{k_1k_2,a_4}^{\phantom{k_1k_2,}\mathcal{G}}(\cdot)+D^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3,k_1k_2k_3}(\cdot)\nu_{k_1k_2k_3,a_4}^{\phantom{k_1k_2k_3,}\mathcal{G}}(\cdot)\\
&\quad+D^{\mathcal{G}\,\mathcal{G}}_{a_1a_2,k_1k_2,k_3k_4}(\cdot)\nu_{k_1k_2,a_3}^{\phantom{k_1k_2,}\mathcal{G}}(\cdot)\nu_{k_3k_4,a_4}^{\phantom{k_3k_4,}\mathcal{G}}(\cdot)\}\\
&=D^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3a_4}(\cdot)+\frac{\langle 4\rangle}{4}\{D^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3,k_1k_2}(\cdot)\nu_{k_1k_2,a_4}^{\phantom{k_1k_2,}\mathcal{G}}(\cdot)+D^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3,k_1k_2k_3}(\cdot)\nu_{k_1k_2k_3,a_4}^{\phantom{k_1k_2k_3,}\mathcal{G}}(\cdot)\}\\
&\quad+\frac{\langle 6\rangle}{6}\,D^{\mathcal{G}\,\mathcal{G}}_{a_1a_2,k_1k_2,k_3k_4}(\cdot)\nu_{k_1k_2,a_3}^{\phantom{k_1k_2,}\mathcal{G}}(\cdot)\nu_{k_3k_4,a_4}^{\phantom{k_3k_4,}\mathcal{G}}(\cdot)\,,
\end{aligned}$$

which may vary from one test statistic in $\mathcal{T}_N$ to another, where $\sum_{\{a_1\ldots a_R\}}$ stands for the summation over $R!$ permutations of $\{a_1,\ldots,a_R\}$. The meanings of these notations are apparent by rewriting

$$\widetilde{Z}^{(N)}_{j_1\cdots j_R}=(\widetilde{Z}^{(N)}_{j_1\cdots j_R}-\widetilde{\nu}_{j_1\ldots j_R,b_{R+1}}^{\phantom{j_1\ldots j_R,b_{R+1}}\mathcal{G}}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot 2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_{R+1}})+\widetilde{\nu}_{j_1\ldots j_R,b_{R+1}}^{\phantom{j_1\ldots j_R,b_{R+1}}\mathcal{G}}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot 2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_{R+1}}\,,\qquad R=2,3$$

in (3). Also, it is customary to define

$$\begin{aligned}
\mathcal{M}_{j_1j_2,j_3j_4}(\cdot)&=\nu_{j_1j_2,j_3j_4}(\cdot)-\nu_{j_1j_2,k}(\cdot)\nu^{k,k'}(\cdot)\nu_{j_3j_4,k'}(\cdot)\,,\\
\mathcal{N}_{j_1j_2,j_3,j_4}(\cdot)&=\nu_{j_1j_2,j_3,j_4}(\cdot)-\nu_{j_1j_2,k}(\cdot)\nu^{k,k'}(\cdot)\nu_{j_3,j_4,k'}(\cdot)\,,
\end{aligned}$$

corresponding to $Cov^{(N)}_\theta(Z^{\perp(N)}_{j_1j_2}(\boldsymbol{\theta}),Z^{\perp(N)}_{j_3j_4}(\boldsymbol{\theta}))$ and $N^{1/2}Cum^{(N)}_\theta(Z^{\perp(N)}_{j_1j_2}(\boldsymbol{\theta}),Z^{(N)}_{j_3}(\boldsymbol{\theta}),Z^{(N)}_{j_4}(\boldsymbol{\theta}))$, where

$$Z^{\perp(N)}_{j_1\cdots j_R}(\boldsymbol{\theta})=Z^{(N)}_{j_1\cdots j_R}(\boldsymbol{\theta})-\nu_{j_1\ldots j_R,k}(\boldsymbol{\theta})\nu^{k,k'}(\boldsymbol{\theta})Z^{(N)}_{k'}(\boldsymbol{\theta})\,.$$

For our derivation, it is helpful to use the null covariances

$$Cov^{(N)}_\theta(Z^{0(N)}_a(\boldsymbol{\theta}),Z^{(N)}_r(\boldsymbol{\theta}))=Cov^{(N)}_\theta(Z^{\perp(N)}_{j_1\cdots j_v}(\boldsymbol{\theta}),Z^{(N)}_j(\boldsymbol{\theta}))\equiv 0$$

(i.e., the variance matrix of $[Z^{0(N)}_a(\boldsymbol{\theta})]_{a=1,\ldots,p_1}$, $[Z^{(N)}_r(\boldsymbol{\theta})]_{r=p_1+1,\ldots,p}$ and $[Z^{\perp(N)}_{jj'}(\boldsymbol{\theta}),Z^{\perp(N)}_{jj'j''}(\boldsymbol{\theta})]_{j,j',j''\in\{1,\ldots,p\}}$ under $P^{(N)}_\theta$ is block diagonal), where

$$Z^{0(N)}_a(\boldsymbol{\theta})=Z^{(N)}_a(\boldsymbol{\theta})-\nu_{a,s}(\boldsymbol{\theta})\nu^{s,s'}_{(2,2)}(\boldsymbol{\theta})Z^{(N)}_{s'}(\boldsymbol{\theta})=\mathcal{G}_{j,a}(\boldsymbol{\theta})Z^{(N)}_j(\boldsymbol{\theta})\,.$$

**Proposition 1 (Kakizawa (2012b))** *The Cordeiro-Ferrari adjustment for $T^{(N)}\in\mathcal{T}_N$ is given by*

$$T^{\mathrm{CF}(N)}=\Big[1-\frac{2}{N}\Big\{\frac{\widetilde{\beta}^C_3}{p_1(p_1+2)(p_1+4)}(T^{(N)})^2+\frac{\widetilde{\beta}^{CD}_2}{p_1(p_1+2)}T^{(N)}+\frac{\widetilde{\beta}^{CD}_1}{p_1}\Big\}\Big]T^{(N)}\tag{4}$$

(*the author has obtained the closed-form expressions for $\beta^{CD}_1(\cdot)=-\Gamma^{CD}_{b_1b_2}(\cdot)\nu^{b_1,b_2}_{(11\cdot 2)}(\cdot)$, $\beta^{CD}_2(\cdot)$ and $\beta^C_3(\cdot)$*).

It is well known (see Introduction) that the LR test statistic $\mathrm{LR}^{(N)} = 2(\widehat{\mathcal{L}}^{(N)} - \widetilde{\mathcal{L}}^{(N)})$ is Bartlett correctable. Proposition 1 indicates that Rao's test statistic $\mathrm{R}^{(N)} = (\widetilde{\mathbf{Z}}^{(N)}_{(1)})'\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot 2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}$ is also Bartlett correctable if $\nu_{j_1,j_2,j_3}(\cdot) = \nu_{j_1,j_2,j_3,j_4}(\cdot) \equiv 0$, $j_1, j_2, j_3, j_4 \in \{1, \dots, p\}$ (in that case, $\beta_2^{\mathrm{R}}(\cdot) = \beta_3^{\mathrm{R}}(\cdot) \equiv 0$).

## 3. Generalized Cordeiro-Ferrari adjustment

Our primary goal is to propose, as the alternative to $T^{\mathrm{CF}(N)}$ (see (4)), an improved test statistic of the form

$$T^{\mathrm{GCF}(N)} = T^{(N)} + \frac{2}{N} \sum_{R=2,4,6} \widetilde{\Gamma}_{b_1 \dots b_R} \prod_{i=1}^{R} [\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot 2)} \widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i} \,, \tag{5}$$

where functions $\Gamma_{a_1 \dots a_R}(\cdot)$'s are of class $\mathcal{C}^1(\boldsymbol{\Theta})$. Without loss of generality, we assume that for $R = 2, 4, 6$, $\Gamma_{a_1 \dots a_R}(\cdot)$ is symmetric under permutation of $\{a_1, \dots, a_R\}$. In what follows, we will give necessary and sufficient conditions on $\Gamma_R(\cdot) = [\Gamma_{a_1 \dots a_R}(\cdot)]_{a_1, \dots, a_R \in \{1, \dots, p_1\}}$, $R = 2, 4, 6$, such that

$$P^{(N)}_{\theta^\dagger}[T^{\mathrm{GCF}(N)} \le x] = \Pr[\chi^2_{p_1} \le x] + \mathrm{o}(N^{-1})\,.$$

Recall that

$$T^{(N)} = U^{(N)}_a \nu^{a,b}_{(11\cdot 2)} U^{(N)}_b + \frac{1}{N^{3/2}}\, \mathrm{o}^{(N)}_{\theta^\dagger}(1 + \min(\delta/2, \xi), \max(5/2, \beta))\,,$$

where

$$U^{(N)}_a = Z^{0(N)}_a + \frac{1}{N^{1/2}}\, U^{C(N)}_a + \frac{1}{N}\, U^{CD(N)}_a\,, \quad a = 1, \dots, p_1$$

are polynomials in $[Z^{0(N)}_a]_{a=1,\dots,p_1}$, $[Z^{(N)}_r]_{r=p_1+1,\dots,p}$ and $[Z^{\perp(N)}_{jj'}, Z^{\perp(N)}_{jj'j''}]_{j,j',j'' \in \{1,\dots,p\}}$, as shown in Kakizawa (2012b). Then, we first show that (5) admits the stochastic expansion

$$T^{\mathrm{GCF}(N)} = U^{\Gamma_{2,4,6}(N)}_a \nu^{a,b}_{(11\cdot 2)} U^{\Gamma_{2,4,6}(N)}_b + \frac{1}{N^{3/2}}\, \mathrm{o}^{(N)}_{\theta^\dagger}(1 + \min(\delta/2, \xi), \max(5/2, \beta)) \tag{6}$$

with

$$U^{\Gamma_{2,4,6}(N)}_a = U^{(N)}_a + \frac{1}{N} \sum_{R=2,4,6} \Gamma_{b_1 \dots b_{R-1} a} \prod_{i=1}^{R-1} [\boldsymbol{\nu}^{-1}_{(11\cdot 2)} \mathbf{Z}^{(N)}_{(1)}]_{b_i}\,, \quad a = 1, \dots, p_1\,.$$

We next compute the null cumulants (up to $\mathrm{o}(N^{-1})$)

$$\begin{aligned}
E^{(N)}_{\theta^\dagger}[U^{\Gamma_{2,4,6}(N)}_{a_1}] &= \frac{\kappa^C_{a_1}}{N^{1/2}} + \mathrm{o}(N^{-1})\,,\\
Cov^{(N)}_{\theta^\dagger}(U^{\Gamma_{2,4,6}(N)}_{a_1}, U^{\Gamma_{2,4,6}(N)}_{a_2}) &= \nu_{(11\cdot 2)a_1,a_2} + \frac{\kappa^{CD}_{a_1,a_2}}{N} + \frac{\langle 2\rangle}{N}\Big(\Gamma_{a_1a_2} + \langle 3\rangle \Gamma_{bb'a_1a_2}\nu^{b,b'}_{(11\cdot 2)}\\
&\quad + \langle 15\rangle \Gamma_{b_1b_1'b_2b_2'a_1a_2}\nu^{b_1,b_1'}_{(11\cdot 2)}\nu^{b_2,b_2'}_{(11\cdot 2)}\Big) + \mathrm{o}(N^{-1})\,,\\
Cum^{(N)}_{\theta^\dagger}(U^{\Gamma_{2,4,6}(N)}_{a_1}, \dots, U^{\Gamma_{2,4,6}(N)}_{a_3}) &= \frac{\kappa^C_{a_1,a_2,a_3}}{N^{1/2}} + \mathrm{o}(N^{-1})\,,\\
Cum^{(N)}_{\theta^\dagger}(U^{\Gamma_{2,4,6}(N)}_{a_1}, \dots, U^{\Gamma_{2,4,6}(N)}_{a_4}) &= \frac{\kappa^{CD}_{a_1,a_2,a_3,a_4}}{N} + \frac{\langle 4!\rangle}{N}\Big(\Gamma_{a_1a_2a_3a_4} + \langle 10\rangle \Gamma_{bb'a_1a_2a_3a_4}\nu^{b,b'}_{(11\cdot 2)}\Big) + \mathrm{o}(N^{-1})\,,\\
Cum^{(N)}_{\theta^\dagger}(U^{\Gamma_{2,4,6}(N)}_{a_1}, \dots, U^{\Gamma_{2,4,6}(N)}_{a_5}) &= \mathrm{o}(N^{-1})\,,\\
Cum^{(N)}_{\theta^\dagger}(U^{\Gamma_{2,4,6}(N)}_{a_1}, \dots, U^{\Gamma_{2,4,6}(N)}_{a_6}) &= \frac{\langle 6!\rangle}{N}\,\Gamma_{a_1a_2a_3a_4a_5a_6} + \mathrm{o}(N^{-1})
\end{aligned}$$

(the closed-form expressions for $\kappa^C_{a_1}$'s, $\kappa^{CD}_{a_1,a_2}$'s, $\kappa^C_{a_1,a_2,a_3}$'s, $\kappa^{CD}_{a_1,a_2,a_3,a_4}$'s are found in Kakizawa (2012b)). In this way, we can obtain the $N^{-1}$-asymptotic expansion of $P^{(N)}_{\theta^\dagger}[U^{\Gamma_{2,4,6}(N)}_a \nu^{a,b}_{(11\cdot 2)} U^{\Gamma_{2,4,6}(N)}_b \le x]$ via the valid $N^{-1}$-Edgeworth expansion of $(U^{\Gamma_{2,4,6}(N)}_1, \dots, U^{\Gamma_{2,4,6}(N)}_{p_1})'$ (see e.g. Bhattacharya and Ghosh (1978)). By virtue of Chibisov's (1972) lemma applied to (6) (see also Magdalinos (1992)), we finally have

**Theorem 2**

$$P_{\theta^\dagger}^{(N)}[T^{\mathrm{GCF}(N)} \le x] = \int_0^x g_{p_1}(t)\,dt - \frac{2}{N}\,\{\pi_1^{CD\Gamma_2} g_{p_1+2}(x) + \pi_2^{CD\Gamma_4} g_{p_1+4}(x) + \pi_3^{C\Gamma_6} g_{p_1+6}(x)\} + \mathrm{o}(N^{-1})\,,$$

*where*

$$\pi_1^{CD\Gamma_2} = \beta_1^{CD} + \Gamma_{b_1b_2}\nu_{(11\cdot 2)}^{b_1,b_2}\,, \quad \pi_2^{CD\Gamma_4} = \beta_2^{CD} + 3\Gamma_{b_1b_2b_3b_4}\nu_{(11\cdot 2)}^{b_1,b_2}\nu_{(11\cdot 2)}^{b_3,b_4}\,,$$
$$\pi_3^{C\Gamma_6} = \beta_3^{C} + 15\Gamma_{b_1b_2b_3b_4b_5b_6}\nu_{(11\cdot 2)}^{b_1,b_2}\nu_{(11\cdot 2)}^{b_3,b_4}\nu_{(11\cdot 2)}^{b_5,b_6}\,.$$

*In case of $p_1 > 1$, infinitely many functions $[\Gamma_{a_1a_2}(\cdot), \Gamma_{a_1a_2a_3a_4}(\cdot), \Gamma_{a_1a_2a_3a_4a_5a_6}(\cdot)]_{a_1,a_2,a_3,a_4,a_5,a_6\in\{1,\ldots,p_1\}}$ that satisfy $\pi_1^{CD\Gamma_2} = \pi_2^{CD\Gamma_4} = \pi_3^{C\Gamma_6} = 0$ give rise to an improved test statistic.*

**Example 1** The original Cordeiro-Ferrari adjustment (4) is a special case of

$$\Gamma_{a_1a_2}^{\mathrm{CF},CD}(\cdot) = -\frac{\beta_1^{CD}(\cdot)}{\varphi_1}\,\nu_{(11\cdot 2)a_1,a_2}(\cdot)\,, \quad \Gamma_{a_1a_2a_3a_4}^{\mathrm{CF},CD}(\cdot) = -\frac{\beta_2^{CD}(\cdot)}{\varphi_2}\,\frac{\langle 3\rangle}{3}\,\nu_{(11\cdot 2)a_1,a_2}(\cdot)\nu_{(11\cdot 2)a_3,a_4}(\cdot)\,,$$
$$\Gamma_{a_1a_2a_3a_4a_5a_6}^{\mathrm{CF},C}(\cdot) = -\frac{\beta_3^{C}(\cdot)}{\varphi_3}\,\frac{\langle 15\rangle}{15}\,\nu_{(11\cdot 2)a_1,a_2}(\cdot)\nu_{(11\cdot 2)a_3,a_4}(\cdot)\nu_{(11\cdot 2)a_5,a_6}(\cdot)\,, \quad a_1,a_2,a_3,a_4,a_5,a_6 \in \{1,\ldots,p_1\}\,,$$

where $\varphi_1 = p_1$, $\varphi_2 = p_1(p_1+2)$ and $\varphi_1 = p_1(p_1+2)(p_1+4)$.

**Example 2** For any symmetric matrix function $\mathbf{A}(\cdot) = [A_{a_1a_2}(\cdot)]_{a_1,a_2\in\{1,\ldots,p_1\}}$, with $A_{a_1a_2}(\cdot)$'s being of class $\mathcal{C}^1(\boldsymbol{\Theta})$, one may consider[‡‡]

$$\Gamma_{a_1a_2}^{ACD}(\cdot) = -\frac{\beta_1^{CD}(\cdot)}{\phi_1^A(\cdot)}\,A_{a_1a_2}(\cdot)\,, \quad \Gamma_{a_1a_2a_3a_4}^{ACD}(\cdot) = -\frac{\beta_2^{CD}(\cdot)}{\phi_2^A(\cdot)}\,\frac{\langle 3\rangle}{3}\,A_{a_1a_2}(\cdot)A_{a_3a_4}(\cdot)\,,$$
$$\Gamma_{a_1a_2a_3a_4a_5a_6}^{AC}(\cdot) = -\frac{\beta_3^{C}(\cdot)}{\phi_3^A(\cdot)}\,\frac{\langle 15\rangle}{15}\,A_{a_1a_2}(\cdot)A_{a_3a_4}(\cdot)A_{a_5a_6}(\cdot)\,, \quad a_1,a_2,a_3,a_4,a_5,a_6 \in \{1,\ldots,p_1\}\,,$$

provided that

$$\phi_1^A(\boldsymbol{\theta}) = \mathrm{tr}\{\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\nu}_{(11\cdot 2)}^{-1}(\boldsymbol{\theta})\} \ne 0\,, \quad \phi_2^A(\boldsymbol{\theta}) = [\mathrm{tr}\{\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\nu}_{(11\cdot 2)}^{-1}(\boldsymbol{\theta})\}]^2 + 2\mathrm{tr}[\{\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\nu}_{(11\cdot 2)}^{-1}(\boldsymbol{\theta})\}^2] \ne 0\,,$$
$$\phi_3^A(\boldsymbol{\theta}) = [\mathrm{tr}\{\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\nu}_{(11\cdot 2)}^{-1}(\boldsymbol{\theta})\}]^3 + 6[\mathrm{tr}\{\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\nu}_{(11\cdot 2)}^{-1}(\boldsymbol{\theta})\}]\mathrm{tr}[\{\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\nu}_{(11\cdot 2)}^{-1}(\boldsymbol{\theta})\}^2] + 8\mathrm{tr}[\{\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\nu}_{(11\cdot 2)}^{-1}(\boldsymbol{\theta})\}^3] \ne 0$$

for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. For examples, we set $\mathbf{A}(\cdot) \equiv \boldsymbol{\gamma}\boldsymbol{\gamma}'$ for any nonzero constant vector $\boldsymbol{\gamma} \in \mathbf{R}^{p_1}$ to define the adjustment

$$T_{\gamma}^{\mathrm{GCF}(N)} = T^{(N)} - \frac{2}{N}\Big\{\widetilde{\beta}_3^{C}\,\frac{(\boldsymbol{\gamma}'\widetilde{\boldsymbol{\nu}}_{(11\cdot 2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)})^6}{15(\boldsymbol{\gamma}'\widetilde{\boldsymbol{\nu}}_{(11\cdot 2)}^{-1}\boldsymbol{\gamma})^3} + \widetilde{\beta}_2^{CD}\,\frac{(\boldsymbol{\gamma}'\widetilde{\boldsymbol{\nu}}_{(11\cdot 2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)})^4}{3(\boldsymbol{\gamma}'\widetilde{\boldsymbol{\nu}}_{(11\cdot 2)}^{-1}\boldsymbol{\gamma})^2} + \widetilde{\beta}_1^{CD}\,\frac{(\boldsymbol{\gamma}'\widetilde{\boldsymbol{\nu}}_{(11\cdot 2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)})^2}{(\boldsymbol{\gamma}'\widetilde{\boldsymbol{\nu}}_{(11\cdot 2)}^{-1}\boldsymbol{\gamma})}\Big\}\,.$$

**Example 3** Instead of the one-parameterization stated above, the equation $\pi_3^{C\Gamma_6} = 0$ has the solution $\Gamma_6(\cdot) = \Gamma_6^{\star C}(\cdot)$ (similar discussions about the solution of $\pi_2^{CD\Gamma_4} = 0$ or $\pi_1^{CD\Gamma_2} = 0$; $\Gamma_4(\cdot) = \Gamma_4^{\star CD}(\cdot)$ and $\Gamma_2(\cdot) = \Gamma_2^{CD}(\cdot) = [\Gamma_{a_1a_2}^{CD}(\cdot)]_{a_1,a_2\in\{1,\ldots,p_1\}}$ are possible), where the elements of $\Gamma_6^{\star C}(\cdot)$ are given by

$$\Gamma_{a_1a_2a_3a_4a_5a_6}^{\star C}(\cdot) = -\frac{1}{6!}\sum_{\{a_1a_2a_3a_4a_5a_6\}}\frac{1}{72}\,\{\nu_{a_1,a_2,a_3}^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}}(\cdot) + 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ a_1a_2a_3}(\cdot)\}\{\nu_{a_4,a_5,a_6}^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}}(\cdot) + 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ a_4a_5a_6}(\cdot)\}\,.$$

The resulting adjustment involves the term $-N^{-1}\{(1/6)(\widetilde{\nu}_{b_1,b_2,b_3}^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}} + 6\widetilde{C}^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ b_1b_2b_3})\prod_{i=1}^3[\widetilde{\boldsymbol{\nu}}_{(11\cdot 2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)}]_{b_i}\}^2$, unlike Kakizawa (2012a,b) with the $N^{-1/2}$-correction $-2N^{-1/2}(1/6)(\widetilde{\nu}_{b_1,b_2,b_3}^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}} + 6\widetilde{C}^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ b_1b_2b_3})\prod_{i=1}^3[\widetilde{\boldsymbol{\nu}}_{(11\cdot 2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)}]_{b_i}$.

[‡‡]In principle, it is possible to use different matrix functions $[{}_vA_{a_1a_2}(\cdot)]_{a_1,a_2\in\{1,\ldots,p_1\}}$, $v = 1,2,3$.

**Remark 1** For the case $\mathrm{LR}^{(N)} = 2(\widehat{\mathcal{L}}^{(N)} - \widetilde{\mathcal{L}}^{(N)})$, it turns out that $\pi_2^{\mathrm{LR},\Gamma_4} = \pi_3^{\mathrm{LR},\Gamma_6} = 0$ by letting $\Gamma_{a_1a_2a_3a_4}(\cdot) = \Gamma_{a_1a_2a_3a_4a_5a_6}(\cdot) \equiv 0$ (note that $\kappa^{\mathrm{LR}}_{a_1,a_2,a_3} = \kappa^{\mathrm{LR}}_{a_1,a_2,a_3,a_4} = 0,\ a_1,a_2,a_3,a_4 \in \{1,\ldots,p_1\}$). Then, the adjustment for the case $\mathbf{A}(\cdot) = \boldsymbol{\nu}_{(11\cdot 2)}(\cdot)$ yields the traditional Bartlett adjustment

$$\Big(1 + \frac{\widetilde{\rho}^{\mathrm{LR}}}{N}\Big)\mathrm{LR}^{(N)} = \mathrm{LR}^{(N)} + \frac{\widetilde{\rho}^{\mathrm{LR}}}{N}\,(\widetilde{\mathbf{Z}}^{(N)}_{(1)})'\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot 2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)} + \frac{1}{N^{3/2}}\,\mathrm{o}^{(N)}_{\theta^\dagger}(1+\delta/2, 3/2)\,,$$

whereas another idea of choosing $\mathbf{A}(\cdot) \equiv \boldsymbol{\gamma}\boldsymbol{\gamma}'$, hence the additive adjustment

$$\mathrm{ALR}^{(N)}_{\gamma} = \mathrm{LR}^{(N)} + p_1\widetilde{\rho}^{\mathrm{LR}}\frac{(\boldsymbol{\gamma}'\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot 2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)})^2}{N\boldsymbol{\gamma}'\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot 2)}\boldsymbol{\gamma}} \quad \text{for any } \boldsymbol{\gamma} \in \mathbf{R}^{p_1} - \{\mathbf{0}_{p_1}\},$$

is found in Kakizawa (2011). Kakizawa (2012b) also gave the other additive adjustments in the form

$$\mathrm{ALR}^{(N)} = \begin{cases} \mathrm{LR}^{(N)} + \dfrac{2}{N}\,(\widetilde{\Gamma}^{\mathrm{LR}}_{b_1b_2} - \widetilde{\Delta}^{\mathrm{LR}}_{b_1b_2})\displaystyle\prod_{i=1}^{2}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot 2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i} \\ \mathrm{LR}^{(N)} + \dfrac{2}{N}\,\widetilde{\Gamma}^{\mathrm{LR}}_{b_1b_2}\displaystyle\prod_{i=1}^{2}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot 2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i}. \end{cases}$$

## 4. Asymptotic expansion of $T^{\mathrm{GCF}(N)}$ under a sequence of local alternatives

It is straightforward (but tedious) to obtain the non-null cumulants of $U_a^{\Gamma_{2,4,6}(N)}$'s up to $\mathrm{o}(N^{-1})$. Then[¶]

**Theorem 3**

$$\begin{aligned} P^{(N)}_{\theta^\dagger + N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[T^{\mathrm{GCF}(N)} > x] &= P_{\theta^\dagger + N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[U_a^{\Gamma_{2,4,6}(N)}\nu^{a,b}_{(11\cdot 2)}U_b^{\Gamma_{2,4,6}(N)} > x] + \mathrm{o}(N^{-1}) \\ &= 1 - G_{p_1}(x; \mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)}) + \sum_{\ell=1}^{2}\frac{2}{N^{\ell/2}}\sum_{v=1}^{3\ell}\dot{\pi}_v^{(\ell)}g_{p_1+2v}(x; \mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)}) \\ &\quad + \mathrm{o}(N^{-1})\,, \end{aligned}$$

*having the forms* $\dot{\pi}_1^{(1)} = \pi_{1[1]}^{(1)} + \pi_{1[3]}^{(1)}$, $\dot{\pi}_2^{(1)} = \pi_{2[1]}^{C(1)} + \pi_{2[3]}^{(1)}$, $\dot{\pi}_3^{(1)} = \pi_{3[3]}^{C(1)}$,

$$\begin{aligned} \dot{\pi}_1^{(2)} &= \pi_1^{CD\Gamma_2} + \pi_{1[2]}^{C(2)} + \pi_{1[4]}^{(2)} + \pi_{1[6]}^{(2)}\,, \\ \dot{\pi}_2^{(2)} &= \pi_2^{CD\Gamma_4} + \{\Gamma_{\diamond\diamond} + \pi_{2[2]}^{CD(2)}\} + \pi_{2[4]}^{C(2)} + \pi_{2[6]}^{(2)}\,, \\ \dot{\pi}_3^{(2)} &= \pi_3^{C\Gamma_6} + \{6\Gamma_{\diamond\diamond bb'}\nu^{b,b'}_{(11\cdot 2)} + \pi_{3[2]}^{CD(2)}\} + \pi_{3[4]}^{C(2)} + \pi_{3[6]}^{C(2)}\,, \\ \dot{\pi}_4^{(2)} &= \{45\Gamma_{\diamond\diamond b_1b'_1b_2b'_2}\nu^{b_1,b'_1}_{(11\cdot 2)}\nu^{b_2,b'_2}_{(11\cdot 2)} + \pi_{4[2]}^{C(2)}\} + \{\Gamma_{\diamond\diamond\diamond\diamond} + \pi_{4[4]}^{CD(2)}\} + \pi_{4[6]}^{C(2)}\,, \\ \dot{\pi}_5^{(2)} &= \{15\Gamma_{\diamond\diamond\diamond\diamond bb'}\nu^{b,b'}_{(11\cdot 2)} + \pi_{5[4]}^{C(2)}\} + \pi_{5[6]}^{C(2)}\,, \\ \dot{\pi}_6^{(2)} &= \Big\{\Gamma_{\diamond\diamond\diamond\diamond\diamond\diamond} + \frac{1}{2}\,(\pi_{3[3]}^{C(1)})^2\Big\} \end{aligned}$$

(*the symbol* $[i]$ *means that the coefficient is a homogeneous polynomial of degree* $i$ *in* $\mathbf{h}_{(1)} \in \mathbf{R}^{p_1}$).

---

[¶] Kakizawa (2012a) obtained

$$\begin{aligned} \dot{\pi}_1^{(1)} &= -\frac{1}{2}\,(\nu_{rr',\diamond}{}^{\mathcal{G}} + \nu_{r,r',\diamond}{}^{\mathcal{G}})\nu^{r,r'}_{(22)} - \frac{1}{6}\,(3\nu^{\mathcal{G}\mathcal{G}\mathcal{G}}_{\diamond\diamond,\diamond} + 2\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\diamond,\diamond,\diamond}) + \frac{1}{2}\,(2\nu^{\;\mathcal{G}\,\mathcal{G}}_{\bullet\diamond,\diamond} + \nu^{\;\mathcal{G}\,\mathcal{G}}_{\bullet,\diamond,\diamond})\,, \\ \dot{\pi}_2^{(1)} &= \frac{1}{2}\,(\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\diamond,b,b'} + 6C^{+\,\mathcal{G}\mathcal{G}\mathcal{G}}_{\;\diamond\, b\, b'})\nu^{b,b'}_{(11\cdot 2)} + \frac{1}{6}\,\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\diamond,\diamond,\diamond}\,, \quad \dot{\pi}_3^{(1)} = \frac{1}{6}\,(\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\diamond,\diamond,\diamond} + 6C^{+\,\mathcal{G}\mathcal{G}\mathcal{G}}_{\;\diamond\diamond\diamond}) \end{aligned}$$

to discuss the $N^{-1/2}$-asymptotic theory under $P^{(N)}_{\theta^\dagger + N^{-1/2}\mathrm{h}}$, where we write $Q_{\ldots\bullet\ldots} = Q_{\ldots j\ldots}h_j$ and $Q_{\ldots\diamond\ldots} = Q_{\ldots a\ldots}h_a$.

Incidentally, if one specializes to the scalar case in the absence of nuisance parameters ($p = p_1 = 1$) and $\Gamma_2 = \Gamma_4 = \Gamma_6 \equiv 0$, the above expressions are in agreement with the findings of Taniguchi (1991b) (see also Rao and Mukerjee (1997)). We also have the asymptotic expansion of

$$P^{(N)}_{\theta^\dagger + N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}\Big[T^{(N)} > \Big\{1+\frac{2}{N}\Big(\frac{\widetilde{\beta}^C_3}{\varphi_3}x^2+\frac{\widetilde{\beta}^{CD}_2}{\varphi_2}x+\frac{\widetilde{\beta}^{CD}_1}{\varphi_1}\Big)\Big\}x\Big] = P^{(N)}_{\theta^\dagger + N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[T^{\mathrm{CF}(N)} > x]+\mathrm{o}(N^{-1})$$

as a special case of Theorem 3 with $\pi_1^{CD\Gamma_2^{\mathrm{CF},CD}} = \pi_2^{CD\Gamma_4^{\mathrm{CF},CD}} = \pi_3^{C\Gamma_6^{\mathrm{CF},C}} = 0$, where

$$\begin{aligned}
\Gamma^{\mathrm{CF},CD}_{\diamond\diamond} &= -\frac{\beta^{CD}_1}{p_1}\left(\mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot2)}\mathbf{h}_{(1)}\right),\\
6\Gamma^{\mathrm{CF},CD}_{\diamond\diamond bb'}\nu^{bb'}_{(11\cdot2)} &= -\frac{2\beta^{CD}_2}{p_1}\left(\mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot2)}\mathbf{h}_{(1)}\right),\\
45\Gamma^{\mathrm{CF},C}_{\diamond\diamond b_1b'_1b_2b'_2}\nu^{b_1b'_1}_{(11\cdot2)}\nu^{b_2b'_2}_{(11\cdot2)} &= -\frac{3\beta^{C}_3}{p_1}\left(\mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot2)}\mathbf{h}_{(1)}\right),\\
\Gamma^{\mathrm{CF},CD}_{\diamond\diamond\diamond\diamond} &= -\frac{\beta^{CD}_2}{p_1(p_1+2)}\left(\mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot2)}\mathbf{h}_{(1)}\right)^2,\\
15\Gamma^{\mathrm{CF},C}_{\diamond\diamond\diamond\diamond bb'}\nu^{bb'}_{(11\cdot2)} &= -\frac{3\beta^{C}_3}{p_1(p_1+2)}\left(\mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot2)}\mathbf{h}_{(1)}\right)^2,\\
\Gamma^{\mathrm{CF},C}_{\diamond\diamond\diamond\diamond\diamond\diamond} &= -\frac{\beta^{C}_3}{p_1(p_1+2)(p_1+4)}\left(\mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot2)}\mathbf{h}_{(1)}\right)^3.
\end{aligned}$$

So, unless $p_1 = 1$, the $N^{-1}$-local power of the Cordeiro-Ferrari adjustment (equivalently the $N^{-1}$-local power of the size adjusted test based on the Cornish-Fisher expansion of the percentile) generally depends on the $C, D$-functions associated with $T^{(N)} \in \mathcal{T}_N$, which contrasts with Rao and Mukerjee (1997).

Using the idea of Example 3, we observe that the $N^{-1}$-local power of the test $T^{\mathrm{GCF}\star(N)} > \chi^2_{p_1}$ is independent of the $D$-functions associated with $T^{(N)} \in \mathcal{T}_N$, where

$$T^{\mathrm{GCF}\star(N)} = T^{(N)} + \frac{2}{N}\Big(\widetilde{\Gamma}^{CD}_{b_1b_2}\prod_{i=1}^{2}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i} + \widetilde{\Gamma}^{\star CD}_{b_1b_2b_3b_4}\prod_{i=1}^{4}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i} + \widetilde{\Gamma}^{\star C}_{b_1b_2b_3b_4b_5b_6}\prod_{i=1}^{6}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i}\Big)$$

or

$$T^{(N)} + \frac{2}{N}\Big(\widetilde{\Gamma}^{CD}_{b_1b_2}\prod_{i=1}^{2}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i} + \widetilde{\Gamma}^{\star CD}_{b_1b_2b_3b_4}\prod_{i=1}^{4}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i} + \widetilde{\Gamma}^{AC}_{b_1b_2b_3b_4b_5b_6}\prod_{i=1}^{6}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i}\Big).$$

**References**


Barndorff-Nielsen, O.E. and Cox, D.R. (1994). *Inference and Asymptotics*. London: Chapman and Hall.

Bartlett, M.S. (1937). Properties of sufficiency and statistical tests. *Proc.Roy.Soc.Lond.* A **160** 268–282.

Bhattacharya, R.N. and Ghosh, J.K. (1978). On the validity of the formal Edgeworth expansion. *Ann.Statist.* **6** 434–451. Correction: (1980). **8** 1399.

Bickel, P.J. and Ghosh, J.K. (1990). A decomposition for the likelihood ratio statistic and the Bartlett correction — A Bayesian argument. *Ann.Statist.* **18** 1070–1090.

Chandra, T.K. and Mukerjee, R. (1991). Bartlett-type modification for Rao's efficient score statistic. *J.Multivariate Anal.* **36** 103–112.

Chibisov, D.M. (1972). An asymptotic expansion for the distribution of a statistic admitting an asymptotic expansion. *Theory Prob.Appl.* **17** 620–630.

Cordeiro, G.M. and Ferrari, S.L.P. (1991). A modified score test statistic having chi-squared distribution to order $n^{-1}$. *Biometrika* **78** 573–582.

Cox, D.R. (1988). Some aspects of conditional and asymptotic inference: A review. *Sankhyā* A **50** 314–337.

Hayakawa, T. (1977). The likelihood ratio criterion and the asymptotic expansion of its distribution. *Ann.Inst.Stat ist.Math.* **29** 359–378. Correction: (1987). **39** 681.

Jensen, J.L. (1993). A historical sketch and some new results on the improved log likelihood ratio statistic. *Scand.J.Statist.* **20** 1–15.

Kakizawa, Y. (1996). Higher order monotone Bartlett-type adjustment for some multivariate test statistics. *Biometrika* **83** 923–927.

Kakizawa, Y. (2009). Third-order power comparisons for a class of tests for multivariate linear hypothesis under general distributions. *J.Multivariate Anal.* **100** 473–496.

Kakizawa, Y. (2010). Comparison of Bartlett-type adjusted tests in the multiparameter case. *J.Multivariate Anal.* **101** 1638–1655.

Kakizawa, Y. (2011). Improved additive adjustments for the LR/ELR test statistics. *Statist.Probab.Lett.* **81** 1245–1255. Correction: **81** 1559.

Kakizawa, Y. (2012a). Second-order powers of a class of tests in the presence of a nuisance parameter. To appear in *Comm.Statist.Theory Methods*.

Kakizawa, Y. (2012b). Improved chi-squared tests for a composite hypothesis. *J.Multivariate Anal.* **107** 141–161.

Lawley, D.N. (1956). A general method for approximating to the distribution of likelihood ratio criteria. *Biometrika* **43** 295–303.

Magdalinos, M.A. (1992). Stochastic expansions and asymptotic approximations. *Econometric Theory* **8** 343–367.

Magdalinos, M.A. (1994). Testing instrument admissibility: Some refined asymptotic results. *Econometrica* **62** 373–403.

Mukerjee, R. (1992). Parametric orthogonality and a Bartlett-type modification for Rao's statistic in the presence of a nuisance parameter. *Statist.Probab.Lett.* **13** 397–400.

Rao, C.R. and Mukerjee, R. (1995). Comparison of Bartlett-type adjustments for the efficient score statistic. *J.Statist.Plann.Inference* **46** 137–146.

Rao, C.R. and Mukerjee, R. (1997). Comparison of LR, score, and Wald tests in a non-iid setting. *J.Multivariate Anal.* **60** 99–110.

Taniguchi, M. (1988). Asymptotic expansions of the distributions of some test statistics for Gaussian ARMA processes. *J.Multivariate Anal.* **27** 494–511.

Taniguchi, M. (1991a). *Higher Order Asymptotic Theory for Time Series Analysis*. Berlin: Springer-Verlag.

Taniguchi, M. (1991b). Third-order asymptotic properties of a class of test statistics under a local alternative. *J.Multivariate Anal.* **37** 223–238.