Abstract

We consider the second order asymptotic properties of an efficient frequency domain regression coefficient estimator $\hat{\beta}$ proposed by Hannan [4]. This estimator is a semiparametric estimator based on nonparametric spectral estimators. We derive the second order Edgeworth expansion of the distribution of $\hat{\beta}$. Then it is shown that the second order asymptotic properties are independent of the bandwidth choice for residual spectral estimator, which implies that $\hat{\beta}$ has the same rate of convergence as in regular parametric estimation. This is a sharp contrast with the general semiparametric estimation theory. We also examine the second order Gaussian efficiency of $\hat{\beta}$. Numerical studies are given to confirm the theoretical results.

AMS 2000 subject classifications: Primary 62E20; 62M10; Secondary 62G20; 62J05

Keywords: Efficient estimation; Second order asymptotics; Semiparametric estimation; Spectral regression

1. Introduction

The problem of efficiently estimating the coefficients in a linear regression model has been investigated widely. When the error covariance matrix depends on unknown parameters, the regression coefficients are often estimated by generalized least squares (GLS), using appropriate consistent estimators of the parameters. It is well known that standardized GLS estimators have the same limiting distribution as the best linear unbiased estimator. Rothenberg [9] gave higher order approximations to the distribution of GLS estimators. Toyooka [13, 14] derived the asymptotic expansion of the mean squared errors (MSE). Since these methods are parametric, standard root $N$ asymptotics hold for time domain GLS estimators, where $N$ is the sample size.

If the autocorrelation structure of the unobservable residuals is not parameterized, we then construct efficient estimators by spectral methods. This technique is semiparametric since it relies on a nonparametric spectral estimator of the residuals.

The semiparametric method of a linear regression model was introduced by Hannan [4], who showed...
that a frequency domain GLS estimator achieves asymptotically the Gauss-Markov efficiency bound under smoothness and Grenander’s conditions on the residual spectral density and the regressor sequence, respectively.

There are principal differences between parametric and nonparametric estimation technique that are often given in terms of consistency and rates of convergence. Velasco and Robinson [15] derived Edgeworth expansions for the distribution of nonparametric estimates. Taniguchi et al. [12] discussed higher order asymptotic theory for minimum contrast estimators of spectral parameters. They established that for semiparametric estimation it does not hold in general that first order efficiency implies second order efficiency.

The semiparametric estimation entails the problem of the bandwidth selection. Applications of higher order asymptotic expansions to this problem have been studied by many authors. Robinson [8] studied frequency domain inference on semiparametric and nonparametric models in the presence of a data dependent bandwidth. Linton [6] investigated the second order properties of various quantities in the partially linear model. Xiao and Phillips [17] gave higher order approximations of the MSE of the frequency domain GLS estimators. Linton and Xiao [7] derived asymptotic expansions for semiparametric adaptive regression estimators. They discussed the bandwidth selection based on minimizing the (integrated) MSE. Also Xiao and Phillips [18] discussed higher order approximations for Wald statistics in frequency domain regressions with integrated processes.

Taniguchi et al. [10] established the root $N$ asymptotic theory for functionals of nonparametric spectral density estimators. This is due to the fact that integration of nonparametric spectral density estimators recovers root $N$ consistency. Since the Hannan estimator is based on integral functionals of nonparametric estimators, it may be expected that the Hannan estimator has attractive properties in higher order asymptotic theory.

In this paper, we will develop the second order asymptotic theory for the frequency domain GLS estimator proposed by Hannan [4]. First, we give the second order Edgeworth expansion of the distribution of the Hannan estimator. Next, we show that the bias-adjusted version of the Hannan estimator is not second order asymptotically Gaussian efficient in general. Of course, if the residual is Gaussian, it is second order asymptotically efficient. As in Xiao and Phillips [17], if the error is a Gaussian process, then it holds that first order efficiency implies second order efficiency.

An interesting result of the paper is that the second order asymptotic properties are independent of the bandwidth choice for the residual spectral estimator. This implies that the Hannan estimator has the same rate of convergence as in regular parametric estimation. This is a sharp contrast with the general semiparametric estimation theory, where it is known that the second order asymptotic properties are strongly influenced by the bandwidth (e.g., Taniguchi et al. [12]).

The paper is organized as follows. Section 2 gives the basic assumptions entertained in the paper. Section 3 gives a number of preliminary results and the main results on the second order Edgeworth expansions. Section 4 contains the discussion on Gaussian efficiency. Proofs are relegated to the Appendix.

2. The model

We consider the following linear regression model

$$y(t) = B'x(t) + u(t), \quad t = 1, \ldots, N,$$

(1)

where $x(t) = (x_1(t), \ldots, x_q(t))'$ is a known vector and nonrandom design sequence, $B = [\beta_{jk}]$ is a $(q \times p)$-matrix of unknown regression parameters, and $u(t) = (u_1(t), \ldots, u_p(t))'$ is an unobserved stationary residual.

The vector process $\{u(t)\}$ is supposed to satisfy the following assumption
(A1) \( \{ u(t) \} \) is a linear process generated by

\[
u(t) = \sum_{s=-\infty}^{\infty} A(s) \epsilon(t-s),
\]

where \( \epsilon(t) = (\epsilon_1(t), \ldots, \epsilon_r(t))' \) are independent identically distributed random vectors with \( \mathbb{E}[\epsilon(t)] = 0, \mathbb{E}[\epsilon(t)\epsilon(t)'] = G \) and finite absolute moments.

(A2) The \((p \times r)\)-matrices \( A(s), s = 0, \pm 1, \ldots \), satisfy

\[
\sum_{s=-\infty}^{\infty} (1 + |s|^2) \|A(s)\| < \infty,
\]

where \( \|A\| \) is the square root of the greatest eigenvalue of \( A^*A \) and \( A^* \) is the conjugate transpose of a matrix \( A \).

Then \( \{ u(t) \} \) has the spectral density matrix

\[
F(\lambda) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \Gamma(s) e^{-i s \lambda},
\]

where \( \Gamma(s) = \mathbb{E}[u(t)u(t+s)'] \).

(A3) There exists a positive constant \( \gamma_1 \) such that

\[
\det \{ F(\lambda) \} \geq \gamma_1 > 0
\]

for \( \lambda \in (-\pi, \pi] \).

**Remark 2.1.** The conditions (A1) and (A2) are satisfied by a wide class of time series models which contains the usual VARMA processes. Under (A1) and (A2), the joint \( k \)-th order cumulants of \( u_j(s), u_{j_2}(s+s_1), \ldots, u_{j_k}(s+s_{k-1}) \)

\[
\Gamma_{j_1 \ldots j_k}(s_1, \ldots, s_{k-1}) = \text{cum}^{(k)}[u_j(s), u_{j_2}(s+s_1), \ldots, u_{j_k}(s+s_{k-1})]
\]

exist and satisfy

\[
\sum_{s_1, \ldots, s_{k-1} = -\infty}^{\infty} (1 + |s|^2) \|\Gamma_{j_1 \ldots j_k}(s_1, \ldots, s_{k-1})\| < \infty, \quad j_1, \ldots, j_k = 1, \ldots, p
\]

for \( l = 1, \ldots, k-1 \). Then \( \{ u(t) \} \) has the \( k \)-th order cumulant spectral density

\[
F_{j_1 \ldots j_k}(\lambda_1, \ldots, \lambda_{k-1}) = \left( \frac{1}{2\pi} \right)^{k-1} \sum_{s_1, \ldots, s_{k-1} = -\infty}^{\infty} \Gamma_{j_1 \ldots j_k}(s_1, \ldots, s_{k-1}) e^{-i(s_1 \lambda_1 + \cdots + s_{k-1} \lambda_{k-1})}.
\]

(A1)-(A3) imply that \( F(\lambda)^{-1} \) exists and has the Fourier series representation

\[
F(\lambda)^{-1} = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \Delta(s) e^{i s \lambda}, \quad \sum_{s=-\infty}^{\infty} (1 + |s|^2) \|\Delta(s)\| < \infty.
\]

This follows from an application of a famous theorem due to Wiener (see, for example, [16, Section 12]).
Let $d_j(N)$ be the positive square root of $\sum_{t=1}^{N} \{x_j(t)\}^2$ for $j = 1, \ldots, q$ and 
\[ D_N = \text{diag}\{d_1(N), \ldots, d_q(N)\}. \]

We impose some assumptions on \{\(x(t)\)\}.

(G1) \(\{x(t)\}\) is uniformly bounded; that is, there exists a positive constant \(\gamma_2\) such that 
\[ \sup_{t \in \mathbb{Z}} |x_j(t)| < \gamma_2, \quad j = 1, \ldots, q. \]

(G2) There exists \(\gamma_3 > 0\) such that \(d_j(N)^2 \geq \gamma_3 N\) for \(j = 1, \ldots, q\).

(G3) There exist \(\eta_j\) such that 
\[ N \sum_{t=1}^{N} \frac{x_j(t)}{d_j(N)} = N^{1/2} \eta_j + O(N^{-1/2}), \quad j = 1, \ldots, q. \]

(G4) There exist regression spectral measures \(M_{j_1 \ldots j_k}(\lambda_1, \ldots, \lambda_{k-1})\) such that 
\[ \sum_{t=1}^{N} \frac{x_{j_1}(t)x_{j_2}(t+l_1) \cdots x_{j_k}(t+l_k-1)}{d_{j_1}(N) \cdots d_{j_k}(N)} \]
\[ = N^{-k/2+1} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{i(l_1 \lambda_1 + \cdots + (k-1) \lambda_{k-1})} dM_{j_1 \ldots j_k}(\lambda_1, \ldots, \lambda_{k-1}) \]
\[ + O(N^{-k/2}) \]
for \(k = 2, 3, \ldots\).

(G5) \(R(0)\) is nonsingular. Here \(R(0)\) is the \((q \times q)\)-matrix given by 
\[ R(l) = \int_{-\pi}^{\pi} e^{i\lambda l} dM(\lambda), \quad l = 0, \pm 1, \ldots, \]

where \(M(\lambda) = [M_{jk}(\lambda)]\).

**Remark 2.2.** Conditions (G1)-(G5) is a higher order version for Grenander’s conditions. For example, linear combinations of harmonic functions satisfy conditions (G1)-(G5). Let us consider a example of \(\eta_j\) and \(M_{j_1 \ldots j_k}(\lambda_1, \ldots, \lambda_{k-1})\).

**Example 2.1 (Harmonic trend).** Suppose \(x_j(t) = \cos \nu_j t, \ j = 1, \ldots, q\), where \(0 < \nu_1 < \cdots < \nu_q < \pi\). From the relation 
\[ \sum_{t=1}^{N} \cos \nu t = \frac{1}{2} \left\{ \frac{\sin(N + 1/2)\nu}{\sin \nu/2} - 1 \right\}, \quad \nu \neq 0, \pm 2\pi, \ldots, \]

it is seen that 
\[ \sum_{t=1}^{N} \frac{x_j(t)}{d_j(N)} = \frac{1}{\sqrt{2}} N^{-1/2} \left\{ \frac{\sin(N + 1/2)\nu_j}{\sin \nu_j/2} - 1 \right\} + O(N^{-3/2}), \]
which means \(\eta_j = 0\). It is well known that \(M(\lambda)\) has a jump \(\text{diag}(0, \ldots, 0, 1/2, 0, \ldots, 0)\) \((1/2\) is in the \(j\)-th diagonal) at \(\lambda = \pm \nu_j\).
To construct the Hannan estimator, we use the spectral window $W_N(\cdot)$ and the lag window $w(\cdot)$ which satisfy the following assumption:

(W1) The function $W_N(\cdot)$ can be expanded as

$$W_N(\lambda) = \frac{1}{2\pi} \sum_{l=-M}^{M} w\left(\frac{l}{M}\right) e^{-i\lambda l}.$$  

(W2) $w(x)$ is a continuous, even function with $w(0) = 1$ and $w(x) = 0$ for $|x| \geq 1$, and satisfies

$$|w(x)| \leq 1,$$

$$\lim_{x \to 0} \frac{1 - w(x)}{|x|^2} < \infty.$$  

(W3) $M = M(N)$ satisfies

$$M/N^{1/3} + N^{1/4}/M \to 0$$  

as $N \to \infty$.

**Remark 2.3.** It is easy to see that the Tukey-Hanning window and Parzen window satisfy (W1) and (W2) (see Hannan [5, pp. 278-279]).

As in Hannan [4], we define for two sequences $y(t)$ and $x(t)$ of $N$ scalars

$$\hat{F}_{yx}(\lambda) = \frac{1}{2\pi N} \sum_{l=-M}^{M} w\left(\frac{l}{M}\right) \sum_{m=1}^{N-1} y(m)x(m + l)e^{-i\lambda l},$$

where $l = \max(0, -l)$ and $l = \max(0, l)$ for $l \in \mathbb{Z}$.

This serves to define all such functions as

$$\hat{F}_{uy}(\lambda), \quad \hat{F}_{xz}(\lambda), \quad \hat{F}_{ux}(\lambda), \quad \hat{F}_{yx}(\lambda), \quad \hat{F}_{ux}(\lambda).$$

We also use the matrix notation

$$\hat{F}_{yy}(\lambda) = [\hat{F}_{uy}(\lambda)], \quad \hat{F}_{xz}(\lambda) = [\hat{F}_{xz}(\lambda)], \quad \hat{F}_{uu}(\lambda) = [\hat{F}_{ux}(\lambda)],$$

$$\hat{F}_{yx}(\lambda) = [\hat{F}_{yx}(\lambda)], \quad \hat{F}_{ux}(\lambda) = [\hat{F}_{ux}(\lambda)].$$

It is not assumed that all of them are estimates of well defined spectral density matrices. Indeed $\hat{F}_{uu}(\lambda)$ is constructed from the actual $u(t)$ and not estimates of them.

We consider a frequency domain version of (1), viz.

$$\hat{F}_{yx}(\lambda) = B^\prime \hat{F}_{xz}(\lambda) + \hat{F}_{ux}(\lambda),$$

which we rewrite in the tensor notation

$$\hat{f}_{yx}(\lambda) = \{I_p \otimes \hat{F}_{xz}(\lambda)\}^\prime \beta + \hat{f}_{ux}(\lambda),$$

where $\hat{f}_{yx}(\lambda) = \text{vec}[\hat{F}_{yx}(\lambda)'], \hat{f}_{ux}(\lambda) = \text{vec}[\hat{F}_{ux}(\lambda)'], \beta = \text{vec}[B]$, and $I_p$ is the $(p \times p)$ identity matrix.

The Hannan estimator of $\beta$ in an integration version is given by

$$\hat{\beta} = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{F}_{uu}(\lambda)^{-1} \otimes \hat{F}_{xz}(\lambda)'d\lambda\right]^{-1} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \{\hat{F}_{uu}(\lambda) \otimes I_q\}^{-1} \hat{f}_{yx}(\lambda)d\lambda\right].$$  

(2)
Since the actual $u(t)$ is unobservable, the quantity $\tilde{F}_{uu}(\lambda)$ is infeasible. Therefore, we use $\tilde{F}_{uu}(\lambda)$ for the estimate of $F(\lambda)$ obtained from the residuals, $\tilde{u}(t) = y(t) - \tilde{B}_{LS}x(t)$, from the least squares regression. Then $\tilde{F}_{uu}(\lambda)$ can be calculated directly as

$$
\tilde{F}_{uu}(\lambda) = \tilde{F}_{yy}(\lambda) - \tilde{F}_{yx}(\lambda)\tilde{B}_{LS} = \tilde{B}_{LS}^{t}\tilde{F}_{yy}(\lambda) + \tilde{B}_{LS}^{t}\tilde{F}_{xx}(\lambda)\tilde{B}_{LS}.
$$

Under very general conditions, $\hat{\beta}$ is first order asymptotically Gaussian efficient; that is, the distribution of $(I_p \otimes D_N)(\hat{\beta} - \beta)$ converges as $N \to \infty$ to the multivariate normal distribution with zero mean vector and covariance matrix given by

$$
\mathcal{I}^{-1} = \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\lambda)^{-1} \otimes dM(\lambda) \right]^{-1}.
$$

3. Second order asymptotic theory

It is well known that integration of nonparametric estimators recovers root $N$ consistency (cf. Taniguchi et al. [10]). Since $\beta$ in (2) is based on integral functionals of nonparametric estimators, it may be expected that $\hat{\beta}$ has attractive properties in higher order asymptotic theory. Thus we consider the second order asymptotic properties of the estimator $\hat{\beta}$. First, we give the following theorem.

**Theorem 3.1.** The stochastic expansion for $(I_p \otimes D_N)(\hat{\beta} - \beta)$ is given by

$$
(I_p \otimes D_N)(\hat{\beta} - \beta) = \mathcal{I}^{-1}Z_1 - N^{-1/2}\mathcal{I}^{-1}(Z_2 - E[Z_2]) - N^{-1/2}\mathcal{I}^{-1}E[Z_2] + N^{-1/2}\mathcal{I}^{-1}Z_3\mathcal{I}^{-1}Z_1 + o_p(N^{-1/2}),
$$

where

$$
Z_1 = \frac{N}{2\pi} \int_{-\pi}^{\pi} \{ F(\lambda)^{-1} \otimes D_N^{-1} \} \hat{f}_{uu}(\lambda)d\lambda,
$$

$$
Z_2 = \frac{N^{3/2}}{2\pi} \int_{-\pi}^{\pi} \{ F(\lambda)^{-1}V_1(\lambda)F(\lambda)^{-1} \otimes D_N^{-1} \} \hat{f}_{ux}(\lambda)d\lambda,
$$

$$
Z_3 = \frac{N^{3/2}}{2\pi} \int_{-\pi}^{\pi} \{ F(\lambda)^{-1}V_1(\lambda)F(\lambda)^{-1} \} \otimes \{ D_N^{-1}\hat{F}_{xx}(\lambda)'D_N^{-1} \}d\lambda,
$$

$$
V_1(\lambda) = \hat{F}_{uu}(\lambda) - E[\hat{F}_{uu}(\lambda)].
$$

Next, we evaluate the asymptotic cumulants of $Z_j$, $j = 1, 2, 3$ given in Theorem 3.1. Denote by $Z_1(jk)$ and $Z_2(jk)$ the $(j-1)q + k$-th component of the vectors $Z_1$ and $Z_2$, respectively. Similarly, denote by $Z_3(j_1 j_2 k_1 k_2)$ the $((j_1 - 1)q + k_1, (j_2 - 1)q + k_2)$-th element of the matrix $Z_3$. Then we have the following lemma.
Lemma 3.1.

\[ E[Z_1] = 0, \]
\[ E[Z_2(jk)] = \sum_{j_1,j_2=1}^{p} K_{j_1j_2j_2}(0,0)F_{j_1j_2}(0)\eta_k + o(1), \]
\[ E[Z_3] = 0, \]
\[ \text{Cov}[Z_1] = I + o(N^{-1/2}), \]
\[ \text{Cov}[Z_1, Z_2] = O(M/N^{1/2}), \]
\[ \text{Cov}[Z_1(j_1k_1), Z_2(j_2k_2, j_3k_3)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_{j_1j_2j_3}(\lambda, -\lambda)\eta_k dM_{k_2k_3}(\lambda) + o(1), \]
\[ \text{cum}[Z_1(j_1k_1), Z_1(j_2k_2), Z_3(j_3k_3)] = N^{-1/2} \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K_{j_1j_2j_3}(\lambda_1, \lambda_2) dM_{k_1k_2k_3}(\lambda_1, \lambda_2) + o(N^{-1/2}), \]

where

\[ K_{jk}(\lambda_1, \lambda_2) = F^{j\prime}(\lambda_1 - \lambda_2)F^{k\prime}(\lambda_1)F^{l\prime}(\lambda_2)F^{m\prime}(\lambda_1), \]

and \( F^{jk}(\lambda) \) is the \((j, k)\)-th element of the matrix \( F(\lambda)^{-1} \). Here we use the Einstein summation convention.

Denote by \( I_{j_1k_1, j_2k_2} \) the \(((j_1 - 1)q + k_1, (j_2 - 1)q + k_2)\)-th element of the matrix \( I \). From Theorem 3.1 and Lemma 3.1 the asymptotic cumulants of \((I_p \otimes D_N)(\hat{\beta} - \beta)_{jk} = d_{k}(N)(\hat{\beta}_{k} - \beta_{k})\) are evaluated as follows:

\[ E[(I_p \otimes D_N)(\hat{\beta} - \beta)_{jk}] = -N^{-1/2} I_{j_1k_1, j_2k_2} \sum_{j_2, j_3=1}^{p} K_{j_1j_2j_3}(0,0)F_{j_2j_3}(0)\eta_k, \]
\[ + N^{-1/2} \frac{1}{2\pi} I_{j_1k_1, j_2k_2} I_{j_2k_2, j_3k_3} \int_{-\pi}^{\pi} K_{j_3j_1j_2}(\lambda_1 - \lambda_2)\eta_k dM_{k_1k_2}(\lambda) + o(N^{-1/2}), \]
\[ = N^{-1/2} C^{jk} + o(N^{-1/2}), \]

(say),

\[ \text{Cov}[(I_p \otimes D_N)(\hat{\beta} - \beta)_{j_1k_1}, (I_p \otimes D_N)(\hat{\beta} - \beta)_{j_2k_2}] = I_{j_1k_1, j_2k_2} + o(N^{-1/2}), \]
\[ \text{cum}[(I_p \otimes D_N)(\hat{\beta} - \beta)_{j_1k_1}, (I_p \otimes D_N)(\hat{\beta} - \beta)_{j_2k_2}, (I_p \otimes D_N)(\hat{\beta} - \beta)_{j_3k_3}] = N^{-1/2} \frac{1}{2\pi} I_{j_1k_1, j_2k_2} I_{j_2k_2, j_3k_3} I_{j_3k_3, j_4k_4} \times \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K_{j_4j_1j_2}(\lambda_1, \lambda_2) dM_{k_1k_2k_3}(\lambda_1, \lambda_2) + o(N^{-1/2}), \]
\[ = N^{-1/2} C^{j_1k_1, j_2k_2, j_3k_3} + o(N^{-1/2}), \]

(say).
The $L$-th order cumulants of $(I_p \otimes D_N)(\hat{\beta} - \beta)_{jk}$ satisfy
\[
\text{cum}^{(L)}[(I_p \otimes D_N)(\hat{\beta} - \beta)_{j_1 k_1}, \ldots, (I_p \otimes D_N)(\hat{\beta} - \beta)_{j_L k_L}] = O(N^{-L/2+1})
\]
for each $L \geq 3$.

From the general Edgeworth expansion formula (e.g., Taniguchi and Kakizawa [11, pp. 169]) we get the following theorem.

**Theorem 3.2.**
\[
\Pr[(I_p \otimes D_N)(\hat{\beta} - \beta) \leq z] = \int_{-\infty}^{z} N(w : \mathbf{I}^{-1}) \left[ 1 + N^{-1/2} C^{jk} H_{jk}(w) \right] \, dw + o(N^{-1/2}),
\]
where $z$ and $w$ are the pq-vectors with $z_{jk}$ and $w_{jk}$ in $(j-1)q + k$-th place, respectively,
\[
N(w : \mathbf{I}^{-1}) = (2\pi)^{-pq/2} |\mathbf{I}|^{1/2} \exp \left( -\frac{1}{2} w' \mathbf{I} w \right),
\]
the multivariate normal distribution, and multivariate Hermite polynomials:
\[
H_{j_1 k_1, \ldots, j_s k_s}(w) = (-1)^s N(w : \mathbf{I}^{-1}) \frac{\partial^s}{\partial w_{j_1 k_1} \cdots \partial w_{j_s k_s}} N(w : \mathbf{I}^{-1}).
\]

The preceding results are unexpected.

**Remark 3.1.** In the context of semiparametric estimation, it is known that root-$N$ asymptotics in general do not hold (e.g., Taniguchi et al. [12]). However, our results claim that, in a linear regression model, standard root-$N$ asymptotics hold up to second order. This means that the Hannan estimator has the same rate of convergence as regular parametric estimation. Moreover, it is seen that our Edgeworth expansion is independent of the bandwidth and the window type function for the residual spectra. This is in sharp contrast with the general semiparametric estimation theory.

We examine the performance of the second order Edgeworth expansion given in Theorem 3.2. The model used for data generation is the following:
\[
y(t) = \beta x(t) + u(t), \quad (p = q = 1)
\]
\[
u(t) = a u(t - 1) + \varepsilon(t),
\]
where $|a| < 1$, $\varepsilon(t)$’s are i.i.d. $\text{Exp}(0, 1)$ random variables with probability density
\[
p(z) = \exp\{- (z + 1)\}, \quad z > -1.
\]

In the following Figure 1-4, we plotted of the first (solid) and the second (dotted) order approximation, and empirical distribution (dashes) which is obtained by 10000 times replications.
Figure 1: $a = 0.5$ and $x(t) = 1$. 
Figure 2: $a = 0.75$ and $x(t) = 1$.

Figure 3: $a = 0.25$ and $x(t) = 1 + \cos t$. 
From Figure 1-4, we observed that the second order Edgeworth expansions are quite accurate in the neighborhood of $z = 0$.

4. Efficiency

In this section we discuss higher order asymptotic efficiency of the Hannan estimator $\hat{\beta}$ defined by (2). To discuss higher order efficiency and establish unified higher order results we need to restrict the class of estimators to second order asymptotically median unbiased (AMU).

From theorem 3.2, it can be seen that $\hat{\beta}$ is not second order AMU. Thus we modify $\hat{\beta}$ as follows:

$$
\hat{\beta}^{*j} = \hat{\beta}^{jk} - N^{-1/2}(I_p \otimes D_N)^{-1}\tilde{C}^{jk} \\
+ \frac{1}{6} N^{-1/2}(I_p \otimes D_N)^{-1}\left(\tilde{\mathcal{I}}^{jk, hk} - \tilde{\mathcal{C}}^{jk, jk, jk}\right),
$$

where

$$
\tilde{\mathcal{I}} = \frac{N}{2\pi} \int_{-\pi}^{\pi} \tilde{F}_{uu}(\lambda)^{-1} \otimes \left\{ D_N^{-1} \tilde{F}_{xx}(\lambda)' D_N^{-1} \right\} d\lambda,
$$

and, $\tilde{C}^{jk}$ and $\tilde{C}^{jk, jk, jk}$ are the quantities replacing the cumulant spectrum by the nonparametric spectral estimator in $C^{jk}$ and $C^{jk, jk, jk}$, respectively.

Then we have the following theorem

**Theorem 4.1.** (i) The estimator $\hat{\beta}^{*j}$ is second order AMU.
The second order asymptotic distribution of \( \hat{\beta}^{jk} \) is

\[
\Pr[(I_p \otimes D_N)(\hat{\beta} - \beta) \leq z] = \int_{-\infty}^{z} N(w : \mathcal{I}^{-1}) \left[ 1 + \frac{1}{6} N^{-1/2} C^{jk,jk,jk} H_{jk}(w) \right. \\
+ \left. \frac{1}{6} N^{-1/2} C^{j1,j2,k2,j3} H_{j1,k1,j2,k2}(w) \right] dw \\
+ o(N^{-1/2}).
\]

Since \( \hat{\beta} \) is first order asymptotically efficient under Gaussian errors, we concentrate our attention only the Gaussian efficiency. From Akahira and Takeuchi [1], the second order Gaussian efficient bound distribution of \( jk \)-component is given by

\[
\Pr[d_j(N)(\hat{\beta}_{jk} - \beta_{jk}) \leq z] = \Phi((\mathcal{I}^{-1})^{jk})^{-1/2} z + o(N^{-1/2}),
\]

where \( \mathcal{I}_{jk} \) is \((j_1,k_1,j_2,k_2)\)-component of the covariance matrix \( \mathcal{I}^{-1} \) of the best linear unbiased estimator. Hence, we have the following result.

**Theorem 4.2.** The bias-corrected estimator \( \hat{\beta}^* \) is second order asymptotically Gaussian efficient, if and only if

\[
C^{jk,jk,jk} = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K_{jjj}(\lambda_1, \lambda_2) dM_{kkk}(\lambda_1, \lambda_2) = 0.
\]

**Remark 4.1.** If the residual \( \{u(t)\} \) is a Gaussian process, then (3) holds. However, in general, the bias-corrected estimator \( \hat{\beta}^* \) is not second order asymptotically Gaussian efficient.

**Remark 4.2.** Theorem 4.1 can be employed to check whether the Hannan estimator leads to a second order Gaussian efficient estimator. Since we do not assume the normality of the error process, in general we have \( K_{jjj}(\lambda_1, \lambda_2) \neq 0 \). Here, we give four examples of the regressor \( \{x(t)\} \) in the case where \( p = q = 1 \).

(i) \( x_1(t) = 1 \) for \( t = 1, 2, \ldots \). Then \( \eta_1 = 1 \), \( M_{11}(\lambda) \) has the jump 1 at \( \lambda = 0 \) and \( M_{11}(\lambda_1, \lambda_2) \) has the jump 1 at \( \lambda_1 = \lambda_2 = 0 \). Hence, the Hannan estimator is second order Gaussian efficient if and only if \( F_{11}(0) = 0 \).

(ii) \( x_1(t) = \cos \nu t, \nu \in (0, 2\pi/3) \) for \( t = 1, 2, \ldots \). Then \( M_{11}(\lambda_1, \lambda_2) \) has the jump \( O_p(N^{-3/2}) \). Hence, the Hannan estimator is always second order Gaussian efficient.

(iii) \( x_1(t) = 1 + \cos \nu t \) for \( t = 1, 2, \ldots \). Then \( \eta_1 = (2/3)^{1/2}, M_{11}(\lambda) \) has the jump 2/3 and 1/6 at \( \lambda = 0 \) and \( \lambda = \pm \nu \), respectively, and \( M_{111}(\lambda_1, \lambda_2) \) has the jump \( (2/3)^{3/2} \) and \( (2/3)^{3/2}/2 \) at \( \lambda_1 = \lambda_2 = 0 \) and \( (\lambda_1, \lambda_2) = (0, \pm \nu), (\pm \nu, 0), (\nu, -\nu), (-\nu, \nu) \), respectively. Hence, the Hannan estimator is not second order Gaussian efficient.

(iv) \( x_1(t) = t/N \) for \( t = 1, 2, \ldots \). Then \( \eta_1 = \sqrt{3}/2 \), \( M_{11}(\lambda) \) has the jump 1 at \( \lambda = 0 \) and \( M_{111}(\lambda_1, \lambda_2) \) has the jump \( 3^{3/2}/4 \) at \( \lambda_1 = \lambda_2 = 0 \). Hence, the Hannan estimator is second order Gaussian efficient if and only if \( F_{111}(0) = 0 \).

**Acknowledgments**

The author would like to express his sincere thanks to Professor Masanobu Taniguchi for his encouragement and guidance.
References


