Some distributional problems associated with non-stationary time series

Katsuto Tanaka
Hitotsubashi University, Tokyo, Japan

Abstract

I discussed some computational problems associated with distributions of statistics arising from nonstationary time series. In particular, I dealt with distributions of (ratios of) quadratic functionals of the ordinary Brownian motion (Bm) and the fractional Bm (fBm). As far as the ordinary Bm is concerned, quadratic functionals were earlier suggested by several authors as test statistics for goodness of fit tests. Ratios of such functionals have been used in time series-based econometrics in connection with unit root tests and cointegration analysis. Here I first demonstrated how to derive and compute the distributions of such functionals by using various examples. Then I illustrated some computational difficulties and indicated a few unsolved problems to be solved.

1. Quadratic functionals of the ordinary Bm

I consider the statistic S which takes the following form:

$$S = \int_0^1 \int_0^1 K(s,t)\, dW(s)dW(t), \tag{1}$$

where it is assumed that

(a) The kernel $K(s,t)$ is symmetric, continuous and positive definite.

(b) $\{W(t)\}$ is the standard Brownian motion defined on $[0,1]$.

The statistic S was earlier suggested as test statistics for goodness of fit, whereas it emerged in time series-based econometrics as weak limits of the normalized second moments associated with a unit root process.

2. Derivation of the characteristic function

To derive the distribution of the statistic S in (1), the following theorem is very useful.

Theorem (Anderson-Darling 1952, Hochstadt 1973)

It holds that

$$\begin{aligned} \mathrm{E}(e^{i\theta S}) &= \mathrm{E}\left[\exp\left\{i\theta \int_0^1 \int_0^1 K(s,t)\, dW(s)dW(t)\right\}\right] \\ &= \prod_{n=1}^{\infty} \left(1 - \frac{2i\theta}{\lambda_n}\right)^{-1/2} = (D(2i\theta))^{-1/2}, \end{aligned}$$

where λ_n is the eigenvalue of K repeated as many times as its multiplicity and $D(\lambda)$ is the Fredholm determinant (FD) of K.

We can derive the FD by considering a differential equation with boundary conditions equivalent to the original integral equation. See, for details, Nabeya and Tanaka (1988) and Tanaka (1990, 1996).

3. Numerical inversion of the characteristic function (c.f.)

Now we can compute the distribution of S in (1) by using the formula

$$F(x) = P(S \le x) = \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{1 - e^{-i\theta x}}{i\theta} \phi(\theta) \right] d\theta,$$

where $\phi(\theta) = (D(2i\theta))^{-1/2}$ is the c.f. of S, and $D(\lambda)$ is the FD of K.

Numerical integration such as Simpson's rule may be used, but care must be taken in computing $\phi(\theta)$ because the computation of square roots of complex-valued functions is involved, which any computer cannot do properly.

4. Some unsolved problems

Lest us consider the type II fBm defined by

$$W_\delta(t) = \frac{1}{\Gamma(1+\delta)} \int_0^t (t-u)^\delta \, dW(u), \tag{2}$$

where δ is a differencing parameter which takes any value greater than $-1/2$. Our interest is how to derive the c.f. of the following quantity:

$$S_\delta = \int_0^1 W_\delta^2(t)\, dt = \int_0^1 \int_0^1 K_\delta(s,t)\, dW(s) dW(t), \tag{3}$$

where

$$K_\delta(s,t) = \frac{1}{\Gamma^2(1+\delta)} \int_0^{\min(s,t)} \{(u-s)(u-t)\}^\delta \, du.$$

We also have another unsolved problem related to the above. Let us consider the process

$$y_j = \rho y_{j-1} + v_j, \ (1-L)^\delta v_j = \varepsilon_j, \ \rho = 1, \ \{\varepsilon_j\} \sim \text{i.i.d.}(0, \sigma^2), \tag{4}$$

where δ is any positive number. The process $\{y_j\}$ is called an I$(1+\delta)$ process - a process with integration order $1+\delta$. It holds that

$$T(\hat{\rho} - 1) \ \Rightarrow \ R(\delta) = \frac{\frac{1}{2} W_\delta^2(1)}{\int_0^1 W_\delta^2(t)\, dt} \quad (\delta > 0), \tag{5}$$

where $\hat{\rho}$ is the LSE of ρ. Note that $R(\delta)$ is well defined for $\delta > -1/2$. The problem here is how to compute numerically the distribution of $R(\delta)$ for fractional δ. I have obtained the Fredholm determinant associated with $\delta = 0,\ 1,\ 2$, but not for fractional δ.

In connection with moments of $R(\delta)$, it has been ensured that

$$\mathrm{E}(R(\delta)) \ = \ \mathrm{E}\left[\frac{1}{2} W_\delta^2(1) \Big/ \int_0^1 W_\delta^2(t)\, dt \right] \ = \ 1 + \delta \tag{6}$$

for $\delta = 0$, 1, and 2. It is our conjecture that the relation in (6) holds for any $\delta > -1/2$.