Sum and product of random variables related to skew distributions*

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Abstract

For real world problem, there are many data which can not be modeled by symmetric distributions. Azzalini(1985, 1986) introduced the univariate skew-normal distribution having the probability density function (p.d.f.) of the form

$$2\phi(x)\Phi(\alpha x),\ x, \alpha \in \mathcal{R},$$

where ϕ and Φ are the p.d.f. and cumulative distribution function (c.d.f.) of the standard normal distribution, respectively. For a random variable X, we write $X \sim \mathcal{SN}(\alpha)$, if X has the p.d.f. as given in (1). This class of distributions includes the $\mathcal{N}(0,1)$ distribution and has some properties like the normal and yet is skew. Since then there are many investigating about skew distributions, and more general definitions of skew distributions are also given. Azzalini and Dalla Valle (1996) extended the results to the multivariate setting with the p.d.f. of the form

$$2\phi_p(\mathbf{x}, \Omega)\Phi(\alpha'\mathbf{x}),\quad \mathbf{x}, \alpha \in \mathcal{R}^p,\ \Omega > 0,$$

where $\phi_p(\mathbf{x}, \Omega)$ is the p-dimensional normal p.d.f. with zero mean vector and correlation matrix Ω. As in univariate case, more general definitions are given for the multivariate case in the literature.

In this work, we will present some results about skew distributions. Especially results related to sum and product (ratio) of skew distributions will be given.

First we give a definition about classes of skew distributions.

Definition 1. Let f be a symmetric p.d.f. A random variable (r.v.) X, with p.d.f. f_X, is said to have a skew distribution of f, denote this by $X \sim S(f)$, if $f_X \in S(f)$, where

$$S(f) = \{h | h(x) = 2f(x)G(x),\ x \in \mathcal{R},\ for\ some\ skewing\ function\ G\}. \tag{1}$$

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$S(f)$ is said to be the skew class generated by f, and f_X is said to have a skew p.d.f. of f. Also G is said to be a skewing function, if

$$0 \leq G(x) \leq 1 \quad and \quad G(x) + G(-x) = 1, x \in \mathcal{R}. \tag{2}$$

In Section 2, we will give the distribution of the linear combinations of the order statistics of bivariate r.v.'s which satisfy certain symmetric conditions.

In Section 3, we will discuss some symmetric properties for product of independent random variables. Let $Z = UV$, where U and V are assumed to be independent r.v.'s. As $-(UV) = (-U)V = U(-V)$, Z is symmetric if at least one of U and V is symmetric. On the other hand, Behboodian (1989) gave examples of nonsymmetric independent r.v.'s U and V, such that Z is symmetric. Among others, we will show if both the p.d.f.'s of U and V are symmetric in a neighborhood of 0, the symmetry assumption for Z will imply that at least one of U and V is symmetric. We also have the following result:

Let U and V be independent. Assume UV is symmetric. Then there exist two independent r.v.'s U_1 and V_1, where U_1 is symmetric, and $V_1 \stackrel{d}{=} Y$, such that $U_1 V_1 \stackrel{d}{=} UV$.

Jones (1999) gave a variety of distributional relationships involving a pair (X, Y) of spherically symmetrically distributed r.v.'s. In Section 4, some parallel results related to skew distributions will be presented.

Finally, for non-negative r.v.'s, it is known that negative moments are usually difficult to compute. Recently under the assumption of normality, by using two key lemmas of Meng(2005), Rukhin(2009) provided some formulas relating negative central moments of the quadratic form defined by a positive definite matrix to those determined by the inverse matrix. He also gave similar relationships for ratios of quadratic forms. These results can be used to check the numerical accuracy of different algorithms for evaluation of these moments(see Magnas(1986,1990), and Paolella(2003)). In Section 5, under two multivariate skew normal distributions, we obtain some results parallel to Rukhin(2009).

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