A LAGRANGIAN FAMILY WHICH INCLUDES GENERALIZED NEGATIVE BINOMIAL AND CHARLIER SERIES DISTRIBUTIONS

Tomoaki Imoto
School of Fundamental Science and Technology, Keio University

For two analytic functions $f(t)$ and $g(t)$ in a domain including 0 with $g(0) \neq 0$, the Lagrange expansion (Jensen, 1902; Consul and Famoye, 2005, 1.78, p. 10) is given as a power series

$$f(t) = \sum_{x=0}^{\infty} \frac{u^x}{x!} \{\mathrm{D}^{x-1}[(g(t))^x \mathrm{D} f(t)]\}_{t=0},$$

where $\mathrm{D} = \partial/\partial t$ with $\mathrm{D}^{-1}\mathrm{D} = 1$ and $u = t/g(t)$. General Lagrangian distributions of the first kind (Consul and Shenton, 1972, 1973), denoted by GLD_1, are obtained by using the above equation when t and u are one to one for $|t| \leq 1$, $f(1) = g(1) = 1$ and $\{\mathrm{D}^{x-1}[(g(t))^x \mathrm{D} f(t)]\}_{t=0} \geq 0$ for $x = 0, 1, 2, \ldots$. The probability mass function (pmf) of a random variable X having GLD_1 is given by

$$P(X = x) = \frac{1}{x!} \{\mathrm{D}^{x-1}[(g(t))^x \mathrm{D} f(t)]\}_{t=0}, \quad x = 0, 1, \ldots$$

with probability generating function (pgf) $G(u) = E[X^u] = f(t)$ or $G(u) = f(h^{-1}(u))$, where $u = h(t) = t/g(t)$. As a special case, the generalized negative binomial distribution (GNBD) by Jain and Consul (1971) is obtained from $f(t) = (q + pt)^\nu$ and $g(t) = (q + pt)^\beta$ with the condition $\nu > 0$, $0 < q = 1 - p < 1$ and $p \leq \beta p < 1$ or $\beta = 0$ and ν is a positive integer. Likewise the Geeta distribution (Consul, 1990) is from $f(t) = t$ and $g(t) = [q/(1 - pt)]^{\beta - 1}$ under the same condition for p, q and β except for the case $\beta = 0$ and ν is a positive integer. Many properties and characteristics such as reproductive property, moments, cumulants, and relationship with the queueing theory and modified power series distribution have been studied about GLD_1 (see Consul and Famoye, 2005).

The Charlier series distribution (CSD) by Ong (1988) has pgf

$$G(u) = (q + pu)^N \exp[\lambda(q + pu - 1)]$$

and pmf

$$P(X = x) = \mathrm{e}^{-\lambda p} p^x q^{N-x} L_x^{(N-x)}(-\lambda q), \quad x = 0, 1, \ldots$$

for $0 < p = 1 - q < 1$, $\lambda > 0$ and a positive integer N, where $L_n^{(\alpha)}(z)$ denotes the generalized Laguerre polynomial (Abramowitz and Stegun, 1972, 22.5.54, p. 780) $L_n^{(\alpha)}(z) = \binom{n+\alpha}{n} {}_1F_1(-n; \alpha + 1; z)$. The CSD belongs to GLD_1 through $f(t) = (q + pt)^N \exp[\lambda(q + pt - 1)]$ and $g(t) = 1$. As another example, the non-central negative binomial distribution (NNBD) by Ong and Lee (1979), which arises as a Poisson mixture and a negative binomial mixture, has pgf

$$G(u) = \left(\frac{q}{1 - pu}\right)^\nu \exp\left[\lambda\left(\frac{q}{1 - pu} - 1\right)\right]$$

with $\lambda, \nu > 0$ and $0 < p = 1 - q < 1$. Its pmf is expressible as

$$P(X = x) = \mathrm{e}^{-\lambda p} p^x q^\nu L_x^{(\nu - 1)}(-\lambda q), \quad x = 0, 1, \ldots.$$

As shown later, the NNBD is also obtainable as a member of GLD_1 with $f(t) = (q + pt)^\nu \exp[\lambda(q + pt - 1)]$ and $g(t) = q + pt$.

In the present paper we consider a Lagrangian distribution of the first kind with

$$\begin{cases} f(t) = (q + pt)^\nu \exp\{\lambda(q + pt - 1)\}, \\ g(t) = (q + pt)^\beta \end{cases}$$

for λ, $\nu > 0$, $0 < q = 1 - p < 1$ and $p \leq \beta p < 1$ or $\beta = 0$ and ν is a positive integer. It is clear that the distribution includes GNBD ($\lambda \to 0$), CSD ($\beta = 0$) and NNBD ($\beta = 1$). Various generalizations of CSD and NNBD are known: the generalized non-central negative binomial (Ong and Lee, 1986), four-parameter non-central negative binomial (Ong, 1987), extended non-central negative binomial (Ong and Shimizu, 2009), and generalized Charlier series (Kitano et al., 2005). However, our distribution is different from these.

Its probability mass function is expressed in terms of generalized Laguerre polynomials or, equivalently, a generalized hypergeometric function. The distribution is formulated as a Charlier series distribution generalized by the generalizing Consul distribution and a non-central negative binomial distribution generalized by the generalizing Geeta distribution. We studies about properties of the distribution such as the relationship with queuing theory, mixture, the index of dispersion, recursive formulas and conditional distribution. An illustrative example of data fitting is given.

Key Words: Compound distribution; Consul distribution; Geeta distribution; Lagrangian distribution of the first kind; Non-central negative binomial distribution.

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