A family of distributions on the circle with links to, and applications arising from, Möbius transformation

加藤 昇吾　（統計数理研究所 数理・推論研究系）
M.C. Jones　（The Open University, UK）

1 Introduction

As is the case on the real line, the most familiar distributions on the circle are symmetric unimodal distributions with two parameters, one controlling location (centre of symmetry), the other scale. Among such distributions on the circle, the distributions which have played central roles are the von Mises and wrapped Cauchy distributions. In this article, we propose a four-parameter extension of the von Mises and wrapped Cauchy distributions which allows for both asymmetry and variations in 'tailweight'. The approach to derive our model is one of transformation and the key notion is the important circle to circle function which is the Möbius transformation. An application of the distribution to a regression model is briefly considered.

2 A family of distributions on the circle

Let $\tilde{\Theta}$ be a random variable which follows the von Mises distribution with mean direction 0 and concentration parameter $\kappa\,(\geq 0)$. Apply the Möbius transformation to $\tilde{\Theta}$, namely,

$$e^{i\Theta} = e^{i\mu}\frac{e^{i\tilde{\Theta}} + re^{i\nu}}{re^{i(\tilde{\Theta}-\nu)}+1} \quad \text{or} \quad \Theta = \mu + \nu + 2\arctan[w\tan\{\tfrac{1}{2}(\tilde{\Theta}-\nu)\}],$$

where $0 \leq \mu, \nu < 2\pi$, $0 \leq r < 1$, and $w = (1-r)/(1+r)$. Then we define a new family of distributions on the circle by Θ.

Theorem 1 *The probability density function of* Θ *is given by*

$$f(\theta) = \frac{1-r^2}{2\pi I_0(\kappa)}\exp\left[\frac{\kappa\{\xi\cos(\theta-\eta)-2r\cos\nu\}}{1+r^2-2r\cos(\theta-\gamma)}\right]\frac{1}{1+r^2-2r\cos(\theta-\gamma)}, \quad 0 \leq \theta < 2\pi, \qquad (1)$$

where $\gamma = \mu+\nu$, $\xi = \sqrt{r^4+2r^2\cos(2\nu)+1}$, $\eta = \mu + \arg\{r^2\cos(2\nu)+1+ir^2\sin(2\nu)\}$, *and* $I_0(\kappa)$ *is the modified Bessel function of the first kind and order zero.*

It is clear from the derivation that the distribution (1) includes the von Mises ($r=0$), wrapped Cauchy ($\kappa = 0$) and circular uniform ($r = \kappa = 0$) distributions as special cases. As $\kappa \to 0$ or $r \to 1$, the model converges to a point distribution with singularity at $\theta = \mu + \nu$.

Theorem 2 *The density (1) is symmetric if and only if* $r = 0$, $\nu = 0, \pi$ *or* $\kappa = 0$.

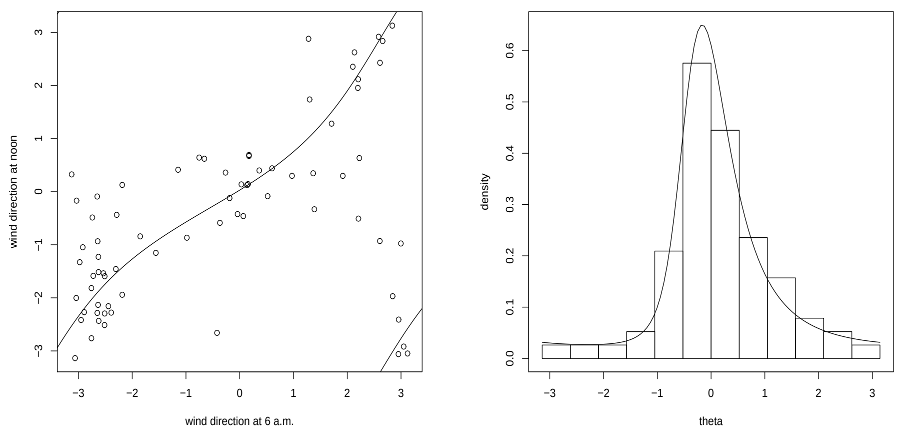
See Kato and Jones (to appear) for some other properties of the presented model.

3 Application to a regression model

We consider a dataset consisting of 73 pairs of wind directions at 6 a.m. and noon measured each day at a particular weather station in Texas, U.S.A. To model this dataset, we use a regression model which adopts the model (1) as an angular error. Let Y_j and x_j $(1 \leq j \leq 73)$ be the response (wind direction at noon) and the covariate (wind direction at 6 a.m.), respectively. Then the regression model we discuss here is given by

$$Y_j = \beta_0 \frac{x_j + \beta_1}{1 + \overline{\beta}_1 x_j} \varepsilon_j, \quad x_j \in U, \tag{2}$$

where $\beta_0 \in U$, $\beta_1 \in \mathbb{C}$, $U = \{z \in \mathbb{C}\,;\, |z| = 1\}$ and $\{\arg(\varepsilon_j)\}$ are independent and distributed as model (1) with circular median 0. This model is an extension of the regression models of Downs and Mardia (2002) and Kato *et al.* (2008), which use the regression curve given in (2) and adopt the von Mises and wrapped Cauchy distributions, respectively, for the angular errors. Here we estimate the parameters based on the method of maximum likelihood.



(a) (b)

Figure 1: (a) planar plot of the wind directions at 6 a.m. and noon with the estimated regression curve, and (b) histogram of the residuals and the fitted density.

Figure 1 shows the planar plot of the wind directions with the estimated regression curve and histogram of the residuals with fitted density. From the figure, it seems that the estimated regression curve provides a reasonable fit to the dataset and the estimated density displays a satisfactory visual fit to the residuals. In addition, according to maximised likelihoods and AIC criterion, the proposed model provides a better fit than the regression models of Downs and Mardia (2002) and Kato *et al.* (2008).

References

Downs, T.D. and Mardia, K.V. (2002). Circular regression. *Biometrika*, **89**, 683–697.

Kato, S. and Jones, M.C. A family of distributions on the circle with links to, and applications arising from, Möbius transformation. *Journal of the American Statistical Association*, to appear.

Kato, S., Shimizu, K. and Shieh, G.S. (2008). A circular–circular regression model. *Statistica Sinica*, **18**, 633–645.