A FAMILY OF DISTRIBUTIONS ON THE UNIT DISC OF GENERAL DIMENSION

Kagumi Uesu and Kunio Shimizu

Research Center for Food Safety, Graduate School of Agricultural and Life Sciences,
The University of Tokyo, 1-1-1 Yayoi, Bunkyo-ku, Tokyo 113-8657, Japan

Department of Mathematics, Keio University,
3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan

Summary

Jones (2004) proposed a new family of distributions on the two dimensional unit disc. These distributions are induced distributions from the bivariate spherical beta distributions using the Möbius transformations, namely, the conformal transformations on the disc, and these distributions are not generally spherically symmetric. We extend this method to the unit discs of general dimension. We obtain a new family of distributions using conformal transformations of the unit disc of general dimension d and the multivariate spherically symmetric beta distributions with joint probability density function (p.d.f.),

(1) $$f_\gamma(y_1,\dots,y_d) = \frac{1}{C_{d,\gamma}}\left(1-\rho_y^2\right)^{\gamma-1},$$

where $\gamma > 0$, $\rho_y^2 = \sum_{i=1}^d y_i^2$ and $C_{d,\gamma} = \frac{\pi^{\frac{d}{2}}\Gamma(\gamma)}{\Gamma(\frac{d}{2}+\gamma)}$ is the normalizing constant. Let $g(a_1,\dots,a_d)$ be a conformal transformation of the d-dimensional unit disc D defined as follows, for $(a_1,\dots,a_d) \neq (0,\dots,0)$,

$$y_i = \frac{x_i(1-\alpha^2) + 2\alpha^2\omega_i\sum_{j=1}^d \omega_j x_j + \omega_i\alpha(1+\rho_x^2)}{1+\alpha^2\rho_x^2 + 2\alpha\sum_{j=1}^d \omega_j x_j}\ (i = 1,2,\dots,d),$$

where $g(a_1,\dots,a_d)(x_1,\dots,x_d) = (y_1,\dots,y_d)$, $r_a = \sqrt{\sum_{i=1}^d a_i^2}$, $\omega_i = a_i/r_a$ and $\alpha = \tanh(r_a/2)$. Then the p.d.f. of the induced distribution of the multivariate spherically symmetric beta distribution (1) by the map $(g(a_1,\dots,a_d))^{-1} = g(-a_1,\dots,-a_d)$ is

$$\frac{1}{C_{d,\gamma}}\Big(\frac{1-\alpha^2}{\alpha^2\rho_x^2 - 2\alpha\sum_{i=1}^d \omega_i x_i + 1}\Big)^{\gamma+d-1}(1-\rho_x^2)^{\gamma-1}\ ,$$

where $\rho_x^2 = \sum_{i=1}^d x_i^2$.

This distribution has the following property.

(1) If $\gamma \neq 1$, then the p.d.f. is unimodal or uni-antimodal.

(2) The Fisher information matrix is positive definite.

(3) The first and the second moments are represented using gamma functions and generalized hyper geometric functions.

(4) The conditional distributions restricted to linear subspaces are other families of distributions on unit discs.

(5) As the value $\sum_{i=1}^d a_i^2$ of the map $g(-a_1, \dots, -a_d)$ goes to infinity, the data of this distribution tend to concentrate to a point on the boundary of D.

There is a measure of color whose values are in the three dimensional disc $D = \{(x, y, z) | x^2 + y^2 + z^2 < 1\}$ where the pair of values (x, y) represents hue and chroma using Munsell color wheel and the value z represents brightness. Thus the data from measurements of colors will be examples of the proposed distribution.

Reference

Jones, M.C., The Möbius distribution on the disc, *Ann. Inst. Statist. Math.*, **56**, 2004; 733-742.