# 最小絶対偏差推定における強固収束

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The least absolute deviation (LAD) estimation is based on the Laplacian loss, as opposed to the least squares estimation on the Gaussian one. As is well known, the LAD estimation is robust to outliers. In a great generality, Knight (1998) formulated asymptotic behavior of the LAD estimator in the context of i.i.d. regression, where the rate of convergence depends on the behavior of the innovation distribution function near the origin. More recently, for infinite-variance autoregressive time series models, Ling (2005) introduced a self-weighted version of the LAD contrast function, entailing asymptotically normally distributed estimators at the usual rate $\sqrt{n}$. It can be expected that, for statistical models of discretely observed stochastic processes, the LAD type estimation can be employed on drift estimation robust to rare and big jumps, which may deteriorate the finite-sample performance of the least squares estimation. Motivated by Ling's paper, we are concerned with the mighty convergence (convergence of moments) as well as an asymptotic normality of self-weighted LAD (SLAD) estimators of discretely observed Ornstein-Uhlenbeck process driven by a Lévy process (OU process).

Let $X = (X_t)_{t\in\mathbb{R}_+}$ be the univariate OU process given by the stochastic differential equation (SDE)

$$dX_t = (\gamma - \lambda X_t)dt + dZ_t$$

with an initial law $\eta := \mathcal{L}(X_0)$, where $Z$ is a nontrivial Lévy process independent of $X_0$; see Masuda (2004) and the references therein for detailed information. We consider estimation of $\theta := (\lambda,\gamma) \in \Theta \subset (0,\infty)\times\mathbb{R}$ based on a discrete-time data $(X_{t_i})_{i=0}^n$ without full specification of $Z$'s Lévy measure. Here $t_i = t_i^n = ih$ with $h = h_n > 0$ such that $h \to 0$ and $nh \to \infty$ as $n \to \infty$, that is, we consider high-frequency and large-time sampling design, which have been well studied for diffusions.

When $Z$ is centered with finite moments and $nh^2 \to 0$ as $nh \to \infty$, the most naive way to achieve our goal would be to use the approximate least squares estimates, which minimizes

$$\theta \mapsto \sum_{i=1}^n \{X_{t_i} - X_{t_{i-1}} - h(\gamma - \lambda X_{t_{i-1}})\}^2.$$

However, the resulting rate of convergence is necessarily $\sqrt{nh}$ (cf. Masuda (2005) for details). Although $\sqrt{nh}$ is known to be optimal in case of diffusions, our main result says that this is far from being optimal in pure-jump cases. Recently, Hu and Long (2009) studied the LSE for $\lambda > 0$ when $Z$ is symmetric $\beta$-stable, with supposing that the true value of $\gamma$ is 0 from the beginning. Our results are completely different from theirs in view of the rate of convergence and the limit distribution.

Our SLAD estimator is defined as a minimizer $\hat{\theta}_n$ of the contrast function

$$\theta \mapsto \sum_{i=1}^n w(X_{t_{i-1}})|X_{t_i} - X_{t_{i-1}} - h(\gamma - \lambda X_{t_{i-1}})|$$

for an appropriate weight function $w$. The usual LAD estimation corresponds to the case where $w \equiv 1$, however, in order also to deduce an asymptotic normality of the SLAD estimators for $Z$ having infinite variance, $w$ should be nontrivial with $w(x) \to 0$ sufficiently fast for

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$|x| \to \infty$. The problem is semiparametric in the sense that the full form of the Lévy measure of $Z$ may be unknown; partly, we impose that the most active part of $Z$ is like symmetric $\beta$-stable for some $\beta \in (0, 2]$. Under regularity conditions, we derive an asymptotic normality of $\hat{\theta}_n$ at rate $\sqrt{n}h^{1-1/\beta}$, where $\beta$ stands for the Blumenthal-Getoor activity index of the driving Lévy process. Therefore, when $Z$ is of pure-jump type (i.e., $\beta < 2$), we have a faster rate of convergence than the familiar $\sqrt{nh}$. The corresponding asymptotic covariance matrix as well as the rate of convergence depends on the unknown index $\beta$, nevertheless, it turns out to be possible to formulate a feasible construction of asymptotic confidence interval. Specifically, we can construct explicit statistics $\hat{T}_n$ such that $\hat{T}_n(\hat{\theta}_n - \theta_0)$ tends to a standard normal distribution, where $\theta_0 = (\lambda_0, \gamma_0) \in \Theta$ denotes the true value of $\theta$.

Also obtained under additional conditions is the mighty convergence of the normalized quantities $\sqrt{n}h^{1-1/\beta}(\hat{\theta}_n - \theta_0)$; for this we apply Yoshida (2005). To the best of author's knowledge, our main result is the first providing the mighty convergence of LAD-type estimates; Ling did not derive such a strong convergence. The mighty convergence is one of crucial tools for studying asymptotic behavior of expected values of statistics depending on estimators, as is the case for the information criteria such as AIC, and also the validity of higher-order asymptotic statistical theory.

Concerning the statistical model in question, we do not currently know what is the phenomenon of asymptotic efficiency: hopefully, we want to derive the local asymptotic normality, but we know that this property cannot hold true at least for the non-Gaussian stable $Z$, as in the case of infinite-variance autoregressive time series models; see Davis *et al.* (1992).

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