RANK OF 3-TENSORS WITH 2 SLICES AND KRONECKER CANONICAL FORM

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1. Introduction

Tensor type data are becoming important recently in various application fields. The factorization of a tensor to a sum of rank 1 tensors means that the data is expressed by a sum of data with simplest structure, and we may have better understanding of data. This is an essential attitude for data analysis and therefore the problem of tensor factorization is an essential one for applications. In this paper we consider the rank problem of 3-tensors with 2 slices. This was studied in the 1970's and 1980's by many authors. JaJa [JA1, JA2] gave the rank for a 3-tensors with 2 slices. He used Kronecker canonical forms of the pencil of two matrices (cf. [G]). Results by Brockett and Dobkin [BD2] are useful for giving a lower bound. JaJa showed that the rank of a Kronecker canonical form without regular pencils is equal to the sum of the ranks of direct summand. However, the rank of a Kronecker canonical form is not equal to the sum of the ranks of direct summand in general and it depends on invariant polynomials. This causes to be difficult to determine the rank of tensors. Our aim is to determine a rank of a tensor T so that $A+T$ is diagonalizable for a given 3-tensor A with 2 slices (see Theorem 3.1), which yields we also obtain the border rank. In this paper we consider ranks of tensors over the complex and real number field.

2. Kronecker canonical form

Any 3-tensor of 2 slices is equivalent to a direct sum each of whose direct summand is

(A) $k \times \ell \times 2$ tensor $(O; O)$,
(B) $k \times k \times 2$ tensor $(\alpha E_k + J_k; E_k)$,
(C) $2k \times 2k \times 2$ tensor $(C_k(c, s) + J_k \otimes E_2; E_{2k})$, $s \neq 0$,
(D) $k \times k \times 2$ tensor $(E_k; J_k)$,
(E) $k \times (k+1) \times 2$ tensor $((\mathbf{0}, E_k); (E_k, \mathbf{0}))$,
(F) $(k+1) \times k \times 2$ tensor $\left(\begin{pmatrix} \mathbf{0}^T \\ E_k \end{pmatrix}; \begin{pmatrix} E_k \\ \mathbf{0}^T \end{pmatrix}\right)$.

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Here $J_k = \begin{pmatrix} 0 & 1 & & O \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$ is a $k \times k$ square matrix and $C_k(c,s) = E_k \otimes \begin{pmatrix} c & -s \\ s & c \end{pmatrix} =$ $\mathrm{Diag}(\begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \dots, \begin{pmatrix} c & -s \\ s & c \end{pmatrix})$ is a $2k \times 2k$ square matrix.

3. Main results

Let A and B be $m \times n$ rectangle matrices. The rank of a tensor $(A; B)$ is obtained by Kronecker canonical form. If $(A; B)$ is equivalent to one consisting of the direct sum of a $m_A \times n_A \times 2$ tensor $(O; O)$ of type (A), an $m_E^{(i)} \times (m_E^{(i)} + 1) \times 2$ tensor of type (E) for $1 \le i \le \ell_E$, and an $(n_F^{(i)} + 1) \times n_F^{(i)} \times 2$ tensor of type (F) for $1 \le i \le \ell_F$, and tensors of type (B), (D) and in addition if $\mathbb{F} = \mathbb{R}$, tensors of type (C). Let α be the maximal integer among the number of $(xE_k + J_k; E_k)$ of type (B) with $k \ge 2$ for each x, the number of $(E_k; J_k)$ of type (D) with $k \ge 2$, and in addition if $\mathbb{F} = \mathbb{R}$ the number of $(C_k(c,s) + J_k \otimes E_j; E_{2k})$ with $k \ge 1$ for each (c,s), $s \ne 0$.

Theorem 3.1. *It holds $m - m_A + \ell_E = n - n_A + \ell_F$ and*

$$\mathrm{rank}_{\mathbb{F}}(A; B) = \alpha + m - m_A + \ell_E.$$

In fact there is a tensor T of rank $\alpha + \ell_E + \ell_F$ such that $(A; B) + T$ is diagonalizable.

Corollary 3.2. *Suppose $m \le n$ and $\mathrm{rank}_{\mathbb{F}}(A; B) = \mathrm{max.rank}_{\mathbb{F}}(m, n, 2)$. Let $X = Y^{\oplus \alpha}$, where Y is $(xE_2 + J_2; E_2)$, $(E_2; J_2)$, or $(C_1(c,s); E_2)$. If n is even, then $(A; B)$ is equivalent to*

$$\mathrm{Diag}(X, ((0,1); (1,0))^{\oplus \ell_E})$$

and otherwise $(A; B)$ is equivalent to one of the following tensors:

- $(\mathrm{Diag}(X, ((0,1); (1,0))^{\oplus \ell_E}), \mathbf{0})$
- $\mathrm{Diag}(X, ((0,1); (1,0))^{\oplus \ell_E}, ((0,1); (1,0))^T)$
- $\mathrm{Diag}(X, ((0,1); (1,0))^{\oplus \ell_E}, (x; 1))$
- $\mathrm{Diag}(X, ((0,1); (1,0))^{\oplus \ell_E}, (1; 0))$
- $\mathrm{Diag}(X, ((0,1); (1,0))^{\oplus (\ell_E - 1)}, ((\mathbf{0}, E_2); (E_2, \mathbf{0})))$

References

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