About the bound of the rank of the set of tensors

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1 Introduction

Parafac is getting now known to be a promising technique of data analysis in many diversified field. Parafac approximates a tensor data, which has more than two indexes, by a sum of more simple structured tensors, rank 1 tensors. The previous work of Sakata-Nishii gave a geometric view of the maximal rank and gave the two concepts of maximal algebraic ranks. They gave a method of calculating the algebraic maximal rank through the Gröbner basis calculation of an elimination ideal corresponding to the problem. One of the concepts of the two algebraic ranks was found to be the same with the usual one of the maximal rank. In this paper we show how to calculate the bound for the maximal rank in three steps. In part A we show an intuitive and elementary method for comparatively small tensors. In part B we consider it for a general case. The part C is a most sophistcated one. Here we give only the summary of the part C for the lack of space.

2 Part C

Let K be a field and let l, m, n be integers such that $1 \leq l \leq m \leq n$. We denote the Segre map by $\varphi\colon K^l \times K^m \times K^n \to K^{lmn}$. Then, we assert a main proposition that with $s = \max(0, \lfloor \frac{l+m-n}{2} \rfloor)$, an arbitrary element of K^{lmn} can be expressed by a sum of

$$lm + sn - sl - sm + s^2 = \min_{0 \leq t \leq l}(lm + tn - tl - tm + t^2)$$

points of $\mathrm{Im}\varphi$. We prove the proposition by induction with respect to l. First we prepare the following notations. The coordinate of K^{lmn} is expressed as $(x_{ijk})_{1 \leq i \leq l, 1 \leq j \leq m, 1 \leq k \leq n}$. For

$$\begin{aligned} \boldsymbol{u} &= (u_1, u_2, \ldots, u_l), \\ \boldsymbol{v} &= (v_1, v_2, \ldots, v_m), \\ \boldsymbol{w} &= (w_1, w_2, \ldots, w_n), \end{aligned}$$

we set

$$\varphi(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) := (u_i v_j w_k).$$

Then we first show the following lemma.

Lemma 2.1 *Any element of K^{lmn} is expressed as a sum of lm elements of* $\mathrm{Im}\varphi$.

Proof. For a given $P = (x_{ijk}) \in K^{lmn}$, setting

$$P_{ij} = \varphi(\boldsymbol{e}_i, \boldsymbol{e}_j, (x_{ij1}, x_{ij2}, \ldots, x_{ijn}))$$

$(1 \leq i \leq l,\ 1 \leq j \leq m)$, we have

$$P = \sum_{i,j} P_{ij}$$

Especially, in the case of $l = 1$ of the induction step, the main proposition clearly holds by this lemma. Now we move to the next induction step for the proof of the main proposition. We assume that $l + m - n \geq 2$. Then we have that $l \geq 2$. For a given (y_{ijk}) in $K^{(l-1)(m-1)n}$, setting

$$x_{ijk} := \begin{cases} y_{i-1,j-1,k} & i > 1,\ j > 1 \\ 0 & i = 1 \text{ or } j = 1 \end{cases},$$

and by identifying (y_{ijk}) as $(x_{ijk}) \in K^{lmn}$, $K^{(l-1)(m-1)n}$ becomes a subspace of K^{lmn}. Now consider $P = (x_{ijk}) \in K^{lmn}$. If necessary, by changing the indexes, we can assume that among x_{11k}, there is an element not equal to 0. Then we choose a basis of K^n, $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, $\ldots$, $\boldsymbol{w}_n$, satisfying

$$(x_{111}, x_{112}, \ldots, x_{11n}) = \boldsymbol{w}_1 + \boldsymbol{w}_2 + \cdots + \boldsymbol{w}_n$$

For (i, j) with $i = 1$ or $j = 1$, putting

$$(x_{ij1}, x_{ij2}, \ldots, x_{ijn}) = \sum_{t=1}^{n} \alpha_{ij}^{(t)} \boldsymbol{w}_t$$

and

$$\begin{aligned} \boldsymbol{u}_t &= (1, \alpha_{21}^{(t)}, \ldots, \alpha_{l1}^{(t)}), \\ \boldsymbol{v}_t &= (1, \alpha_{12}^{(t)}, \ldots, \alpha_{1m}^{(t)}), \\ P_t &= \varphi(\boldsymbol{u}_t, \boldsymbol{v}_t, \boldsymbol{w}_t), \end{aligned}$$

we consider

$$\sum_{t=1}^{n} P_t = (z_{ijk}).$$

Since $\alpha_{11}^{(t)} = 1$ for each t, we see that for $i = 1$ or $j = 1$, $x_{ijk} = z_{ijk}$. Hence, putting

$$Q = P - \sum_{t=1}^{n} P_t,$$

since $Q \in K^{(l-1)(m-1)n}$, by the induction assumption for $l - 1$, the vector Q is expressed as a sum of

$$\begin{aligned} & (l-1)(m-1) + (s-1)n - (s-1)(l-1) - (s-1)(m-1) + (s-1)^2 \\ = \ & lm + (s-1)n - sl - sm + s^2 \end{aligned}$$

elements of $\varphi(K^{l-1} \times K^{m-1} \times K^n)$. From this P is expressed as a sum of

$$lm + (s-1)n - sl - sm + s^2 + n = lm + sn - sl - sm + s^2$$

elements of $\mathrm{Im}\varphi$. This prove the assertion of the proposition.

References

[1] Sakata, T. and Nishii, R. (2006). Algebraic rank analysis of tensor data through Gröbner basis theory, Proc. Computational Statistics, 751–758, Aug. in Roma.