Confidence intervals for diffusions

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Asymptotic expansions of the maximum likelihood estimators (mle) have been thoroughly investigated. Besides thier theoretical consequences such as the higher order efficiency of the mle, they are practically useful in constructing higher order correct intervals as well as those of the likelihood ratio statistics. The formal Edgeworth expansion of the random sequence $\{S_T\}$ with the j-th cumulant $\kappa_{j,T}$ of S_T admitting an expansion

$$\kappa_{j,T} = T^{-(j-2)/2}\left(k_{j,1} + T^{-1}k_{j,2} + T^{-2}k_{j,3} + \dots\right), \;\; k_{1,1} = 0, \;\; k_{2,1} = 1$$

for $j = 1, 2, \dots$ is given as

$$P[S_T \le x] = \Phi(x) + T^{-1/2}p_1(x)\phi(x) + T^{-1}p_2(x)\phi(x) + O(T^{-3/2}),$$

where

$$p_1(x) = -k_{1,2} - k_{3,1}(x^2 - 1)/6$$
$$p_2(x) = -(k_{2,2} + k_{1,2}^2)x/2 - (k_{4,1} + 4k_{1,2}k_{3,1})(x^3 - 3x)/24 - k_{3,1}^2(x^5 - 10x^3 - 15x)/72$$

and Φ, ϕ are the standard normal distribution function and the density respectively. See e.g., Hall [1]. Let $\hat{\theta}_T$ be the mle and $S_T = \sqrt{T}(\hat{\theta}_T - \theta)/\hat{\sigma}_T$ for a true value θ, where $\hat{\sigma}_T^2$ is an estimator of the asymptotic variance of $\sqrt{T}(\hat{\theta}_T - \theta)$. If we consider an iid observation $X_1, \dots, X_T$ of the normal distribution with mean θ and known variance σ^2, then

$$\hat{\theta}_T = \frac{1}{T}\sum_{t=1}^{T} X_t, \;\; \hat{\sigma}_T = \sigma$$

and $P[S_T \le x] = \Phi(x)$. In particular, we have $k_{3,1} = k_{4,1} = 0$ for all θ. We showed that the last simple identity is inherited to normal based models such as the AR processes

$$X_t = \sum_{s=1}^{p} \theta_s X_{t-s} + \epsilon_t, \;\; \epsilon_s \sim \mathrm{NID}(0, \sigma^2)$$

with known σ and the Itô processes

$$dX_t = \sum_{j=1}^{p} \theta_j b_t^j dt + \sigma_t dW_t, \;\; 0 \le t \le T,$$

where W is a standard Brownian motion and b_t, σ_t are known progressively measurable processes. To fix idea, let us concentrate on the one-dimensional ergodic diffusion process

$$dX_t = \theta b(X_t)dt + \sigma(X_t)dW_t.$$

The (conditioal) mle is given as

$$\hat{\theta}_T = \frac{\int_0^T \frac{b(X_t)}{\sigma(X_t)^2} dX_t}{\int_0^T \frac{b(X_t)^2}{\sigma(X_t)^2} dt}.$$

We can use

$$\hat{\sigma}_T = \left\{ \frac{1}{T} \int_0^T \frac{b(X_t)^2}{\sigma(X_t)^2} dt \right\}^{-1/2}$$

as an estimator of the asymptotic variance. See e.g., Kutoyants [2]. One consequence of our results is that

$$P[S_T \le x] = \Phi(x) - T^{-1/2} k_{1,2} \phi(x) + O(T^{-1}) = \Phi(x - k_{1,2}/\sqrt{T}) + O(T^{-1}).$$

The last normal approximation will be useful in constructing second order correct confidence intervals by replacing $k_{1,2}$ with a suitable $\sqrt{T}$-consistent estimator, because this approximation is free from "negative probability" problem which the Cornish-Fisher expansion based argument usually suffers. See e.g., Hall [1].

This curious fact that $k_{3,1} = k_{4,1} = 0$ seems to come from the fact that

$$2(l(\hat{\theta}_T) - l(\theta)) = S_T^2,$$

where l is the log-likelihood, hence the left side sequence is the likelihood ratio statitstic. We discussed our results extending a well-known fact that if

$$2(l(\hat{\theta}_T) - l(\theta)) = \left\{ \sqrt{T}(\hat{\theta}_T - \theta)/\tilde{\sigma}_T \right\}^2 + O_p(T^{-3/2})$$

for some random sequence $\{\tilde{\sigma}_T\}$, then the 3rd and 4th cumulants of $\sqrt{T}(\hat{\theta}_T - \theta)/\tilde{\sigma}_T$ vanish up to $O(T^{-3/2})$.

By the way, for the multi-dimensional diffusion model

$$dX_t^i = b(\theta, X_t)^i dt + c_j^i(X_t) dW_t^j,$$

any reasonable estimator for the asymptotic bias $k_{1,2}$ or the asymptotic skewness $k_{3,1}$ was not available so far. Although the coefficients are presented in Sakamoto and Yoshida [3], they are expressed in terms of the solutions of Laplace equations, whose explicit solutions are not available. We concentrated on the case that the diffusion is symmetric to propose estimators for the coefficients.

参考文献

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[3] Sakamoto, Y.; Yoshida, N. Asymptotic expansion formulas for functionals of ϵ-Markov processes with a mixing property. Ann. Inst. Statist. Math. 56 (2004), no. 3, 545–597.