State Space Models on Special Manifolds

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We consider the state space models assuming the matrix Langevin noise processes on the Stiefel and Grassmann manifolds. The estimation of states via posterior modes is suggested.

The Stiefel manifold $V_{k,m}$ is the space a point of which is a set of k orthonormal vectors in $R^m (k \leq m)$, so that $V_{k,m} = \{X(m \times k); X'X = I_k\}$, where I_k is the $k \times k$ identity matrix. For $m = k, V_{k,m}$ is the orthogonal group $O(m)$ of $m \times m$ orthonormal matrices. A random matrix X on $V_{k,m}$ is said to have the matrix Langevin (or von Mises-Fisher) distribution, denoted by $L(m, k; F)$, if its density function is given by (Downs [2])

$$\exp[\mathrm{tr}(F'X)]/{}_0F_1(\frac{1}{2}m; \frac{1}{4}F'F), \text{ with } F \text{ an } m \times k \text{ matrix},$$

where the ${}_pF_q$ is a hypergemetric function with matrix argument. Here, assuming the rank of F being k for the simplicity of argument, we write the singular value decomposition of F as

$$F = \Gamma\Lambda\Theta', \text{ with } \Gamma \in V_{k,m}, \ \Theta \in O(k), \text{ and } \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_k), \quad \lambda_1 \geq \cdots \geq \lambda_k > 0.$$

which is also expressed as $\Gamma\Theta' \cdot \Theta\Lambda\Theta' = M \cdot C$, say. The distribution has the unique modal orientation $M = \Gamma\Theta' \in V_{k,m}$ and the λ_i's control the concentrations about the mode in the directions determined by the orientations Γ and Θ. These parameters, mode and concentrations, of the matrix Langevin distribution may be considered as the counterparts of the parameters, mean and variance-convariances, of the (multivariate) normal distribution.

The Grassmann manifold $G_{k,m-k}$ is the space whose points are k-planes ν, that is, k-dimensional hyperplanes in R^m containing the origin. To each k-plane ν in $G_{k,m-k}$, corresponds a unique $m \times m$ orthogonal projection matrix P idempotent of rank k onto ν. Let $P_{k,m-k}$ denote the set of all $m \times m$ orthogonal projection matrices idempotent of rank k. We shall conduct our statistical analysis on the manifold $P_{k,m-k}$ which is equivalent to the Grassmann manifold $G_{k,m-k}$.

For the special case $k = 1$, the observations from the unit hypersphere $V_{1,m}$ are directed unit vectors, i.e., directions, and those from the real projective space $G_{1,m-1}$

are axes or undirected lines through the origin, i.e., one-dimensional subspaces. There exists a large literature of applications of these directional statistics and its statistical analysis. The analysis of data on the general Stiefel manifold $V_{k,m}$ is required in particular for $k \leq m \leq 3$ in practical applications in the Earth Sciences, Medical Sciences, Astronomy, Biology, Meteorology, and other fields. See Downs [2], Watson [5], Fisher, Lewis and Embleton [3], and Mardia and Jupp [4]. One is naturally interested in k-dimensional subspaces as observations from the general Grassmann manifold $G_{k,m-k}$. The Grassmann manifold is a rather new subject treated as a statistical sample space. See Chikuse [1] for statistical analyses on the Stiefel and Grassmann manifolds.

We develop a state space model relating the time series observations $\{Y_1, Y_2, \ldots, Y_t\}$ on the Stiefel manifold $V_{k,m}$ to a sequence of unobserved state modal orientation matrices $\{X_0, X_1, \ldots, X_t\}$ on $V_{k,m}$ of noise processes distributed as matrix Langevin. We show a Bayes method for estimating the states $\{X_1, X_2, \ldots, X_t\}$ by the posterior modes assuming X_0 given. An iterative procedure for the estimation is suggested. Further, we consider an extended state space model on Stiefel manifolds, where two sequences of unobserved state modal orientation matrices $\{X_0, X_1, \ldots, X_t\}$ on $V_{k,m}$ and unobserved state regression matrices $\{Y_0, Y_1, \ldots, Y_t\}$ on $O(m)$, in consideration of orientational regressions, are involved.

A simple state space model on the manifold $P_{k,m-k}$, where the time series observations $\{Q_1, Q_2, \ldots, Q_t\}$ on $P_{k,m-k}$ are related to a sequence of unobserved state matrices $\{P_0, P_1, \ldots, P_t\}$ on $P_{k,m-k}$ of noise processes distributed as matrix Langevin.

A more detailed discussion is given in [Y. Chikuse, State Space Models on Special Manifolds, J. Multivariate Anal. 97 (2006) 1284−1294].

References

[1] Y. Chikuse, Statistics on Special Manifolds, Lecture Notes in Statistics, Vol. 174, Springer, New York, 2003.

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[3] N. I. Fisher, T. Lewis, B. J. J. Embleton, Statistical Analysis of Spherical Data, Cambridge University Press, London and New York, 1987.

[4] K. V. Mardia, P. E. Jupp, Directional Statistics, Wiley, New York, 2000.

[5] G. S. Watson, Statistics on Spheres, Lecture Notes in Mathematics (University of Arkansas) 6, Wiley, New York, 1983.