A New Family of Orientational Distributions on Stiefel Manifolds

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In this paper, we concern developing new orientational distributions on the Stiefel manifold. The Stiefel manifold $V_{k,m}$ is the space a point of which is a set of k orthonormal vectors in $R^m (k \leq m)$, so that $V_{k,m} = \{X(m \times k); X'X = I_k\}$, where I_k is the $k \times k$ identity matrix. The manifold $V_{k,m}$ is an analytic manifold of dimension $\frac{1}{2}k(2m-k-1)[= km - \frac{1}{2}k(k+1)]$. It is also a subset of the hypersphere of radius $k^{1/2}$ in R^{km}, since $\mathrm{tr} X'X = k$. For $m = k$, $V_{m,m}$ is the orthogonal group $O(m)$ of $m \times m$ orthonormal matrices. A point of $V_{k,m}$ may also be called an orientation extending the notion of a direction for $k = 1$. The analysis of data on $V_{k,m}(k \geq 1)$ played important roles in the Earth Sciences, Medical Sciences, Astronomy, Biology and many other fields. For discussions of statistical analysis on $V_{k,m}$, see e.g., Watson [4], Fisher, Lewis and Embleton [2], Mardia and Jupp [3], and also Chikuse [1].

We develop orientational distrbutions on the general $V_{k,m}$ employing some methods, based on the matrix-variate t distributions. First, we investigate the method via imposing the condition $Z'Z = I_k$ for an $m \times k$ random matrix Z. Imposing the condition $Z'Z = I_k$ on the matrix-variate t distribution $T_{m,k}(n, M; I_m, \Sigma)$ leads to defining the orientational distribution whose probability density function (p.d.f.) is proportional to $|I_k - F'X|^{-b}$. When Z is distributed as matrix-variate $T_{m,k}(n, 0; \Sigma_1, \Sigma_2)$, we obtain the orientational distribution with p.d.f. proportional to $|I_k - X'BXA|^{-b}$. The normalizing constants of the p.d.f.'s and some properties of these two distributions are obtained.

The next method uses the polar decomposition and its related decomposition of a random matrix. We investgate special cases where the p.d.f.'s thus derived are given by closed mathematical forms. We derive the distribution of the orientation $H_Z = Z(Z'Z)^{-1/2} (\in V_{k,m})$ of an $m \times k$ random matrix Z by integrating the p.d.f. of $Z = H_Z T_Z^{1/2}$ over $T_Z = Z'Z > 0$. We discuss on the distribution of H_Z when Z is distributed as $T_{m,k}(n, 0; \Sigma, I_k)$, which may be called the matrix angular central t (MACT) distribution. Next, we derive the distribution of the reference matrix Y_Z or $P_Z = Y_Z Y_Z' = H_Z H_Z'$, the orthogonal projection matrix of Z, by integrating furthermore the p.d.f. of $H_Z = Y_Z Q_Z$ over $Q_Z \in O(k)$. The distribution derived in this way gives a distribution which is invariant under right-orthogonal transformations on $V_{k,m}$.

When Z is distributed as $T_{m,k}(n, M; I_m, \Sigma)$, we derive the p.d.f. of the distribution of Y_Z or P_Z, which may be called the orthogonal projective t (OPT) distribution.

We consider some inferential problems, e.g., estimation and testing for the parameters, for these distributions developed in this paper. Approximations of the distributions and approximate statistical analyses are concerned.

There are some orientational distributions which have been already defined and discussed for statistical analysis on $V_{k,m}$ in the literature. The distributions which have been most frequently used are the matrix Langevin and the matrix Bingham distributions. The matrix angular central Gaussian and the orthogonal projective Gaussian distributions were also developed as right-orthogonally invariant distributions on $V_{k,m}$. They are known to have been derived based on the matrix-variate normal distributions. It is shown that these distributions are obtainable as limiting distributions of the corresponding distributions developed in this paper.

We suggest a larger family of more generalized distributions whose p.d.f.'s are expressed in terms of hypergeometric functions with matrix argument, and furthermore which have the rotational symmetry around a given subspace. It is seen that most of the distributions discussed in this paper and in the existing literature belong to this larger family.

References

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