Frequency Domain Approach to the Unit Root Analysis

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Abstract

This paper deals with the problem of testing for a unit root in the semiparametric setup of the near-integrated processes proposed by Phillips (Biometrika, 1987). A sequence of transforms is introduced which is called the quarter Fourier transform (QFT) and defined according to the spectral decomposition of the random walk. The transforms $\tilde{z}_k, k = 1, \ldots$ are slightly different from the ordinary finite Fourier transforms. The asymptotic distribution of a finite segment of the sequence which corresponds to the K lowest frequencies is explicitly derived and hence the likelihood function of the semiparametric model is given. The maximum likelihood estimate of α and the t-statistic for α are shown to have the same asymptotic null and alternative distributions as those of the familiar tests such as Phillips' Z_ρ and Z_τ, if we let K slowly to infinity. A consistent estimate of the long run variance is also obtained from the likelihood.
The approach is generalized for the model with unknown level and the model with linear deterministic trend.

Main result of the paper

This paper considers the following model as the DGP:

$$
\begin{aligned}
x_t &= d_t + y_t, \\
y_t &= \rho y_{t-1} + w_t, \qquad t = 1, \ldots, T
\end{aligned}
$$

where d_t is a determinsitic trend term and ρ is localized as $\rho = 1 - \dfrac{\alpha}{T}$ and the error term w_t satisfies the suitable condistions.
A sequence of transform

$$
\tilde{z}_k = \frac{1}{T} \sum_{t=1}^{T} x_t \sqrt{\frac{2}{T+\frac{1}{2}}} \sin \frac{C_k t}{T+\frac{1}{2}} \qquad k = 1, 2, \ldots, \tag{1}
$$

is introduced. The asymptotic distribution of $\tilde{\boldsymbol{z}}_K = \left(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_K\right)'$ is multivariate normal, $\mathrm{N}_K\left(0, \sigma^2 G_K(\alpha)\right)$, where $G_K(\alpha) = diag\{\mathbf{d}_{K0}(\alpha)\} - \alpha(1 + e^{-2\alpha})\, \mathbf{d}_{K1}(\alpha)\, \mathbf{d}'_{K1}(\alpha)$, where $\mathbf{d}_{K0}(\alpha)$

and $\mathbf{d}_{K1}(\alpha)$ the K-dimensional vectors with their k-th entries $1/(\alpha^2 + C_k^2)$ and $(-1)^{k-1}/(\alpha^2 + C_k^2)$, respectively. The log likelihood function is given by

$$L_0(\alpha, \sigma^2) = -\frac{K}{2} \log \sigma^2 - \frac{1}{2} \log |\, G_K(\alpha) \,| - \frac{1}{2\sigma^2} \left(\alpha^2 \tilde{u}_K^2 + K \tilde{s}_K^2 + \frac{\alpha B_K(\alpha)}{2} \tilde{v}_K^2 \right),$$

where $\left(\tilde{u}_K^2, \tilde{v}_K, \tilde{s}_K^2\right) = \left(\sum_{k=1}^{K} \tilde{z}_k^2, \sqrt{2} \sum_{k=1}^{K} (-1)^k \tilde{z}_k, \frac{1}{K} \sum_{k=1}^{K} C_k^2 \tilde{z}_k^2 .\right)$ From this and the fact that $\log G_K(\alpha)/G_K(0) \to \alpha$ and $B_K(\alpha) \to 2$ as $T \to \infty$, we have an estimator $\hat{\alpha}_K = -\dfrac{\left(\tilde{v}_K^2 - \tilde{s}_K^2\right)}{2\tilde{u}_K^2}$. The main result of the paper is that if K_T is such that $K_T \to \infty$ and $K_T/T \to 0$, under suitable conditions on the error term,

$$\hat{\alpha}_{K_T} \xrightarrow{d} -\frac{\displaystyle\int_0^1 J_\alpha(\tau) dJ_\alpha(\tau)}{\displaystyle\int_0^1 J_\alpha(\tau)^2 d\tau},$$

where $J_\alpha(\tau) = \int_0^\tau \exp\left\{-\alpha \left(\tau - \tau'\right)\right\} dW(\tau')$ is the Ornstein-Uhlenbeck process.