Euclid 距離行列の順序構造について

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1 Introduction

The aim of this paper is to clarify the ordering structure of EDMs (Euclidean distance matrices) and spherical EDMs. In Section 2, we introduce a group majorization ordering for EDMs, and then establish two inequalities. The first inequality deals with the radius of a spherical EDM: It states that the radius of a spherical EDM is increasing with respect to the group majorization ordering; The second inequality concerns the spreadth of the eigenvalues of an EDM: It shows that the larger an EDM is in terms of the group majorization ordering, the more spread out its eigenvalues are. Section 3 is devoted to describing minimal elements with respect to this ordering.

2 Ordering for EDMs

Let $\mathbb{G}$ be a compact subgroup of $\mathbb{O}_n$, where $\mathbb{O}_n$ denotes the group of $n \times n$ orthogonal matrices, and let $\mathcal{S}_n$ be the set of $n \times n$ symmetric matrices. The group $\mathbb{G}$ acts on $\mathcal{S}_n$ via the group action

$$\Sigma \to \Gamma\Sigma\Gamma^{\mathrm{T}} \quad \text{with} \quad \Gamma \in \mathbb{G} \text{ and } \Sigma \in \mathcal{S}_n. \tag{2.1}$$

Hence for each $\Sigma \in \mathcal{S}_n$, the $\mathbb{G}$-orbit of Σ is given by $\{\Gamma\Sigma\Gamma^{\mathrm{T}} \mid \Gamma \in \mathbb{G}\}$. For $\Psi, \Sigma \in \mathcal{S}_n$, we write $\Psi \leq_{\mathbb{G}} \Sigma$ if Ψ is in the convex hull of the $\mathbb{G}$-orbit of Σ. Namely,

$$\Psi \leq_{\mathbb{G}} \Sigma \quad \text{iff} \quad \Psi \in \mathrm{co}\{\Gamma\Sigma\Gamma^{\mathrm{T}} \mid \Gamma \in \mathbb{G}\}, \tag{2.2}$$

where, for a set A, the notation co$A$ means the convex hull of A. The ordering $\leq_{\mathbb{G}}$ thus defined is called the group majorization ordering induced by $\mathbb{G}$.

To introduce a group majorization ordering for EDMs, let $\mathbb{G} = \mathbb{P}_n$, where $\mathbb{P}_n$ is the group of $n \times n$ permutation matrices. The group $\mathbb{P}_n$ acts on the set Λ_n via the same action as in (2.1):

$$D \to \Pi D \Pi^{\mathrm{T}} \quad \text{with} \quad \Pi \in \mathbb{P}_n \text{ and } D \in \Lambda_n. \tag{2.3}$$

Let $\tilde{\Lambda}_n$ be the set of all spherical EDMs. Below we often limit our consideration to $\tilde{\Lambda}_n$, and hence the following lemma is helpful, which shows that $\mathbb{P}_n$ acts on the set $\tilde{\Lambda}_n$ via the same action as in (2.3).

Lemma 1. (Kurata and Sakuma (2007)) *For each $D \in \tilde{\Lambda}_n$ and $\Pi \in \mathbb{P}_n$, it holds that $\Pi D \Pi^{\mathrm{T}} \in \tilde{\Lambda}_n$.*

Theorem 1. (Kurata and Sakuma (2007)) *If D_1, $D_2 \in \tilde{\Lambda}_n$ satisfy*

$$D_1 \leq_{\mathbb{P}_n} D_2, \tag{2.4}$$

then the following two inequalities hold:

$$\text{radius}(D_1) \le \text{radius}(D_2) \quad \textit{and} \quad \text{center}(D_1) \le \text{center}(D_2). \tag{2.5}$$

holds, where radius(D) *denote the radius of the sphere of a spherical EDM D, and* center(D) *the length of the center of the sphere.*

Next we state an inequality on the spreadth of the eigenvalues of an EDM $D \in \Lambda_n$. We write $\mathbf{y} \succeq \mathbf{x}$ if $\mathbf{y}$ majorizes $\mathbf{x}$. For $D \in \Lambda_n$, let $\lambda_1(D) \ge \cdots \ge \lambda_n(D)$ be the ordered eigenvalues of D, and let

$$\boldsymbol{\lambda}(D) = (\lambda_1(D), \cdots, \lambda_n(D))^{\mathrm{T}} : n \times 1.$$

Theorem 2. (Kurata and Sakuma (2007), Kurata (2007)) *If $D_1, D_2 \in \Lambda_n$ satisfy $D_1 \le_{\mathbb{P}_n} D_2$, then $\boldsymbol{\lambda}(D_2)$ majorizes $\boldsymbol{\lambda}(D_1)$:*

$$\boldsymbol{\lambda}(D_2) \succeq \boldsymbol{\lambda}(D_1) \quad \textit{and} \quad \boldsymbol{\lambda}(B_2) \succeq \boldsymbol{\lambda}(B_1). \tag{2.6}$$

3 Minimal Elements With Respect To $\le_{\mathbb{P}_n}$

We call an EDM $D \in \Lambda_n$ minimal with respect to the ordering $\le_{\mathbb{P}_n}$, if there is no EDM $\overline{D} \in \Lambda_n$ such that $\overline{D} \ne D$ and $\overline{D} \le_{\mathbb{P}_n} D$.

Theorem 3. (Kurata and Sakuma (2007)) *An EDM $D \in \Lambda_n$ is minimal with respect to the ordering $\le_{\mathbb{P}_n}$ if and only if it is of the form*

$$D = \beta(\mathbf{e}\mathbf{e}^{\mathrm{T}} - I_n) \quad \textit{for some } \beta \ge 0. \tag{3.1}$$

Finally we consider the problem of describing the smallest element of the convex hull of the orbit of a given $D \in \Lambda_n$.

Theorem 4. (Kurata and Sakuma (2007)) *For each $D \in \Lambda_n$ the matrix*

$$m \equiv m(D) = \frac{\mathbf{e}^{\mathrm{T}} D \mathbf{e}}{n(n-1)}(\mathbf{e}\mathbf{e}^{\mathrm{T}} - I_n) \tag{3.2}$$

satisfies

$$m(D) \le_{\mathbb{P}_n} F \quad \textit{for any } F \in \mathrm{co}\{\Pi D \Pi^{\mathrm{T}} \mid \Pi \in \mathbb{P}_n\}.$$

That is, $m(D)$ is the smallest element of $\mathrm{co}\{\Pi D \Pi^{\mathrm{T}} \mid \Pi \in \mathbb{P}_n\}$.

4 Spreadth of Configuration

Fix a spherical EDM $D \in \tilde{\Lambda}_n$ with configuration $\{\mathbf{p}_1, \cdots, \mathbf{p}_n\} \subset \mathbb{R}^r$, and let $-\mathbf{q}$ be the center of $\{\mathbf{p}_1, \cdots, \mathbf{p}_n\}$. Then the vectors

$$\hat{\mathbf{p}}_i = (\mathbf{p}_i + \mathbf{q}) / \|\mathbf{q}\| \quad (i = 1, \cdots, n) \tag{4.3}$$

thus defined are on the unit sphere in $\mathbb{R}^r$. Let SV(D) be the spherical variance of $\{\hat{\mathbf{p}}_1, \cdots, \hat{\mathbf{p}}_n\}$. Then the following monotonicity result holds.

Theorem 5. (Kurata (2007)) *If $D_1, D_2 \in \tilde{\Lambda}_n$ satisfy $D_2 \le_{\mathbb{P}_n} D_1$, then the inequality*

$$\mathrm{SV}(D_2) \le \mathrm{SV}(D_1) \tag{4.4}$$

holds.