Empirical Likelihood Approach for Non-Gaussian Locally Stationary Processes

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1 Introduction

Empirical likelihood method is one of the nonparametric methods for statistical inference proposed by Owen (1988, 1990). In i.i.d. setting, it is shown that empirical likelihood ratio is asymptotically chi-square distributed and used for constructing confidence regions for the mean, a class of M-estimates and so on.

Empirical likelihood method has been applied to many scenes and their applications are also extended to dependent observations. However it seems that they were mainly for stationary processes. Although stationarity is the most fundamental assumption when we are engaged in time series analysis, it is also known that real time series data are generally nonstationary (e.g., economics analysis). Recently Dahlhaus (1996a, 1996b, 1997) proposed an important class of nonstationary processes, called locally stationary processes. Locally stationary processes have so called time varying spectral density whose spectral structures smoothly change in time.

In this paper we extend the empirical likelihood method to non-Gaussian locally stationary processes with time-varying spectral density $g(u, \lambda)$. We derive the asymptotic distribution of empirical likelihood ratio based on the central limit theorem for locally stationary processes, which is seen in Dahlhaus (1997). Especially, when we consider the stationary case, i.e., the time varying spectral density is independent of time parameter u, the asymptotic distribution becomes chi-square.

As an application of this method, we can estimate an extended autocorrelation for locally stationary processes.

2 Setting

We consider an inference problem on a parameter $\boldsymbol{\theta} \in \Theta \subset \mathbf{R}^q$ based on a stretch $X_{1,T}, \ldots, X_{T,T}$ where $\{X_{t,T}\}_{t=1,\ldots,T}$ is a locally stationary process which has the time varying spectral density $g(u, \lambda)$. We suppose that information about $\boldsymbol{\theta}$ exists through the following time-spectral moment condition

$$\int_0^1 \int_{-\pi}^{\pi} \boldsymbol{\phi}(u, \lambda, \boldsymbol{\theta}_0)\, g(u, \lambda)\, d\lambda du = \mathbf{0} \tag{1}$$

where $\boldsymbol{\phi} : [0, 1] \times [-\pi, \pi] \times \mathbf{R}^q \to \mathbf{C}^q$ is an appropriate function and $\boldsymbol{\theta}_0$ is a true value of $\boldsymbol{\theta}$. We give a brief example of ϕ and corresponding θ_0 in scalar case. If we set

$$\phi(u, \lambda, \theta) = \theta - e^{i\lambda k},$$

then (1) leads to

$$\theta_0 = \frac{\int_0^1 \int_{-\pi}^{\pi} e^{i\lambda k} g(u, \lambda)\, d\lambda du}{\int_0^1 \int_{-\pi}^{\pi} g(u, \lambda)\, d\lambda du}. \tag{2}$$

When we consider the stationary case (i.e. $g(u, \lambda)$ is independent of time parameter u), (2) becomes

$$\theta_0 = \frac{\int_{-\pi}^{\pi} e^{i\lambda k} g(\lambda)\, d\lambda}{\int_{-\pi}^{\pi} g(\lambda)\, d\lambda},$$

which expresses the autocorrelation with lag k. So, (2) can be interpreted as a kind of autocorrelation with lag k for locally stationary processes.

3 Main Result

Now we set

$$\boldsymbol{m}_j(\boldsymbol{\theta}) = \int_{-\pi}^{\pi} \boldsymbol{\phi}(u_j, \lambda, \boldsymbol{\theta}) I_N(u_j, \lambda)\, d\lambda \quad (j = 1, \dots, M, \quad M = T - N + 1)$$

as a new estimating function for locally stationary processes. Here $I_N(u, \lambda)$ is a local periodogram with segment length N. Assume that sample size T and segment length N satisfy the following relationship

$$T^{1/4} \ll N \ll T^{1/2} (\log T)^{-1}.$$

We use the following empirical likelihood ratio function $\mathcal{R}(\boldsymbol{\theta})$ defined by

$$\mathcal{R}(\boldsymbol{\theta}) = \max_{(w_1, \dots, w_T)} \left\{ \prod_{j=1}^{M} M w_j \mid \sum_{j=1}^{M} w_j \boldsymbol{m}_j(\boldsymbol{\theta}) = \boldsymbol{0},\ w_j \geq 0,\ \sum_{j=1}^{M} w_j = 1 \right\}. \tag{3}$$

Then we get the following theorem.

Theorem 1 *Suppose $X_{1,T}, \dots, X_{T,T}$ are realization of a locally stationary process. Under some regular conditions*

$$-\frac{1}{\pi} \log \mathcal{R}(\boldsymbol{\theta}_0) \xrightarrow{d} (\boldsymbol{FN})'(\boldsymbol{FN})$$

as $T \to \infty$, where $\boldsymbol{N}$ is a q-dimensional normal random vector with zero mean vector and covariance matrix $\boldsymbol{I}_q$ (identity matrix) and $\boldsymbol{F} = \boldsymbol{\Sigma}_2^{-\frac{1}{2}} \boldsymbol{\Sigma}_1^{\frac{1}{2}}$. Here $\boldsymbol{\Sigma}_1$ is q by q matrix whose (i, j) element is

$$\begin{aligned}(\Sigma_1)_{ij} = \frac{1}{2\pi} \int_0^1 \Bigg[&\int_{-\pi}^{\pi} \phi_i(u, \lambda, \boldsymbol{\theta}_0) \{\phi_j(u, \lambda, \boldsymbol{\theta}_0) + \phi_j(u, -\lambda, \boldsymbol{\theta}_0)\} g(u, \lambda)^2\, d\lambda \\ &+ c_4 \int_{-\pi}^{\pi} \phi_i(u, \lambda, \boldsymbol{\theta}_0) g(u, \lambda)\, d\lambda \int_{-\pi}^{\pi} \phi_j(u, \mu, \boldsymbol{\theta}_0) g(u, \mu)\, d\mu \Bigg] du\end{aligned}$$

and $\boldsymbol{\Sigma}_2$ is q by q matrix whose (i, j) element is

$$\begin{aligned}(\Sigma_2)_{ij} = \frac{1}{2\pi} \int_0^1 \Bigg[&\int_{-\pi}^{\pi} \phi_i(u, \lambda, \boldsymbol{\theta}_0) \{\phi_j(u, \lambda, \boldsymbol{\theta}_0) + \phi_j(u, -\lambda, \boldsymbol{\theta}_0)\} g(u, \lambda)^2\, d\lambda \\ &+ \int_{-\pi}^{\pi} \phi_i(u, \lambda, \boldsymbol{\theta}_0) g(u, \lambda)\, d\lambda \int_{-\pi}^{\pi} \phi_j(u, \mu, \boldsymbol{\theta}_0) g(u, \mu)\, d\mu \Bigg] du.\end{aligned}$$

where c_4 is the fourth order cumulant of an innovation process.

Remark 1 If the process is stationary, that is, the time varying spectral density is independent of time parameter u, we can easily see that $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$ and the asymptotic distribution becomes the chi-square with degree of freedom q.