Lifting between the set of three way contingency tables and r-neibourhood theorem

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1 Lifting by Sasaki's Criation Operator for two way tables

Here we briefly review on creation operators for two way contingency tables. Let $\Omega(\alpha, \beta)$ be the set of all tables with a row sum vector α and a column sum vector β. In the sequential conditional test, whenever a new sample is added in (p, q) cell the basic space of inference changes to $\Omega(\alpha + e_p, \beta + e_q)$ where e_j denotes the j-th canonnical unit vector. We want to generate $\Omega(\alpha + e_p, \beta + e_q)$ from $\Omega(\alpha, \beta)$ where the latter is already in hand. Let $\Phi(\alpha, \beta)$ be a characteristic function of the set $\Omega(\alpha, \beta)$, that is,

$$\Phi(x|\alpha, \beta) = \sum_{H \in \Omega(\alpha,\beta)} \frac{x^a}{a!},$$

where $H = (a_{ij})$ and $x = (x_{ij})$, and $x^a = \prod x_{ij}^{a_{ij}}$ and $a! = \prod_{ij} a_{ij}!$.
Sasaki operators are defined by

$$S_{pq} = x_{pq} + \sum_{i,j} x_{iq} x_{pj} \frac{\partial}{\partial x_{ij}}$$

Then we have the next proposition,
Proposition(Sasaki(1991))

$$S_{pq}\Phi(x|\alpha, \beta) = (1 + \alpha_p)(1 + \beta_q)\Phi(x|\alpha + e_p, \beta + e_q).$$

Our new view of Sasaki operator for $K \times L$ tables is that S_{pq} can be viewed as a set of mappings $\{\delta_{pq}, \phi_{pq}^{ij}, i = 1, 2, .., K \;\; and \;\; j = 1, ..., L\}$ where δ_{pq} denotes a mapping that maps H in $\Omega(\alpha, \beta)$ to $\overline{H} = H +^t e_p e_q$ in $\Omega(\alpha + e_p, \beta + e_q)$ and ϕ_{pq}^{ij} denotes a mapping that maps H into $\overline{H} = H +^t e_i e_q +^t e_p e_j -^t e_p e_q$. Then the above proposition restated that the set of mappings $\mathcal{S} = \{\delta_{11}, \phi_{pq}^{ij}, i = 1, .., K \;\; and \;\; j = 1, .., L\}$ constitutes a surjective set of mappings from $\Omega(\alpha, \beta)$ to $\Omega(\alpha + e_p, \beta + e_q)$. Here it should be noted that the term "surjectivity" is used for some unusual sense. In this paper we call a set $\mathcal{S}$ of mappings, from $\mathcal{X}$ to $\mathcal{Y}$, "surjective" if for any $y \in \mathcal{Y}$ there is some $x \in \mathcal{X}$ and $f \in \mathcal{S}$ such that $y = f(x)$. Further, it is very interesting that Sasaki operator was devised as creation operator of an A-hyperbolic partial differential system and it was first recognized as a creation operator for two way contingency tables in Saito et. al(1997). Sakata and Sawae(2003) used this creation operator in the sequential conditional test and studied the performance of the test.

2 Lifting Operator for Three Way Tables

In this section we consider how to construct a lifting operator for three way contingency tables. Let $\Omega(\alpha, \beta, \gamma)$ be the set of all tables with a fixed $x-y, y-z$ and $x-z$ marginal two way tables, α, β and γ respectively. In the sequential conditional test, whenever a new sample is added in (p, q, r) cell the basic space of inference changes from $\Omega(\alpha, \beta, \gamma)$ to $\Omega(\alpha + E_{pq}, \beta + E_{qr}, \gamma + E_{pr})$ where E_{ij} denotes the two way table with 1 in (i, j) cell and 0 in all other cells. We want to generate $\Omega(\alpha + E_{pq}, \beta + E_{qr}, \gamma + E_{pr})$ from $\Omega(\alpha, \beta, \gamma)$ where the latter is already in hand. That is, we want to obtain $\mathcal{S}$, a set of mappings from $\Omega(\alpha, \beta, \gamma)$ to $\Omega(\alpha + E_{pq}, \beta + E_{qr}, \gamma + E_{pr})$, which is "surjective" in the sense of Section 2 and algorithmically easily implementable. Here we propose a conjecture about a surjective set of mappings. For stating our conjecture, to avoid complexity of suffix, we confine ourselves to the case of $(p, q, r) = (1, 1, 1)$ without loss of generality. We use this notation throughout this paper from here.
The following Theorem 1 is obvious and given without proof, but becomes a powerful tool to prove partially the conjecture in the later section.
Theorem(Ishii and Sakata) All elements $\overline{H}$ with $\overline{H}(1,1,1) \geq 1$ in $\Omega(\alpha+E_{11}, \beta+E_{11}, \gamma+E_{11})$ is obtainable by some H in $\Omega(\alpha, \beta, \gamma)$ by simply applying δ_{111} to H. Let $\mathcal{M}$ be the set of minimal Markov basis and let $\mathcal{M}^k$ be the subset of $\mathcal{M}$ consisting of all Markov moves M with $M_{111} = k$. The following Theorem is one of our target theorems to prove.
Theorem A Assume we have a set of a minimal Markov basis of moves. Let $\phi = \{\delta_{i,j,k}, M_1^0 + M_2^0 + M_3^1\}$ where M_1^0 and M_2^0 are minimal Markov moves with $x_{ijk} = 0$ and M_3^1 is a Markov move with $x_{ijk} = 1$. Then ϕ is a surjective from $\Omega(\alpha, \beta, \gamma)$ to $\Omega(\alpha+\delta_{ij}, \beta+\delta_{jk}, \gamma+\delta_{ik})$ for any α, β, and γ for $3 \times 3 \times 3$ contingency tables.

Let $\mathcal{M}$ be the set of minimal Markov basis. We introduce a r-neibourhood property for $\Omega(\alpha, \beta, \gamma)$.

Definition 2.1 *Let H be an element $\Omega(\alpha, \beta, \gamma)$. An element H' in $\Omega(\alpha, \beta, \gamma)$ is said to be in a r-neighbourhood of H if there is a path of Markov moves $M_1, ..., M_r$ of $\mathcal{M}$ such that*

$$H' = H + M_1 + \cdots + M_r$$

where $H_s = H + M_1 + \cdots + M_s$ is non-negative for all $s \leq r$.

$\Omega(\alpha, \beta, \gamma)$ has *r-neibourhood property* if any element H in $\Omega(\alpha, \beta, \gamma)$ has an element H' with $H'_{ijk} = 1$ within its r-neibourhood. Then Theorem A can be express as
Theorem A' For $3 \times 3 \times 3$ contingency tables, $\Omega(\alpha+\delta_{ij}, \beta+\delta_{jk}, \gamma+\delta_{ik})$ has 2-neibourhood property.

We have the following properties by computer programs.
Theorem B $3 \times 3 \times 4$ contingency tables has 3-neighbourhood property. All proofs are omitted due to the lack of space.