An Approximation of European Option Prices under General Diffusion Processes

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This study proposes a method that allows for fast computation of European option prices under general diffusion processes of the spot price. Fast computation is possible using the Fourier transform (FT) approach to the pricing of options. This approach depends crucially on the availability of analytical expressions for the characteristic function of the log spot price. In reality, however, there are many models proposed in the literature that have no such expression. This study then obtains an analytical approximation of the characteristic function, which makes the FT approach still viable. The computational burden is of almost the same magnitude as that for affine diffusion models having analytical expressions for the characteristic function, while high accuracy is possible.

The FT approach, introduced in option pricing by Heston (1993), first obtains the Fourier transform of the option value, which is then recovered by the inverse transform. This seemingly roundabout approach is powerful and becomes increasingly popular for two reasons. First, realistic modeling is possible, because there are certainly many cases where the option value does not have an analytical expression but its Fourier transform does, allowing for model extension. Second, the computational burden is relatively miner compared with other numerical techniques such as the Monte Carlo (MC) and Finite Difference methods. Although numerical integration is required for the inverse transform, this can be computed quickly using the fast Fourier transform method.

There are, however, many other models proposed in both the time-series and option pricing literatures that do not have analytical expressions for the characteristic function, and hence the FT approach cannot be utilized. One notable example is a model that has both stochastic volatility (SV) and constant elasticity of variance (CEV), as proposed by Jones (2003), and Melino and Turnbull (1990) for pricing options, and by Brenner et al. (1996), and Gallant and Tauchen (1998) for describing the dynamics of the instantaneous risk-free rate. The CEV option pricing model alone has potential to control the volatility level, which tends to be high (low) when the spot price level is low (high). This can generate negative skewness of the conditional return distribution and thus the implied volatility smirk. Furthermore, Jones (2003) reports that a CEV model of the volatility process is appropriate for capturing the observed behavior, which tends to be intensive when the volatility level is high. A more descriptive option pricing model, therefore, seems to have SV with CEV for both the spot price and volatility processes. However, analytical expressions for the characteristic function are generally unavailable. Another example is a model with a stochastic risk-free rate. In a theoretical viewpoint, the stochastic nature of the risk-free rate is endogenously derived from equilibrium models. Also, in a practical viewpoint, the risk-free rate needs to be stochastic when its variation strongly affects the variation in the spot price. In this case, modeling the instantaneous correlation between the spot price and the risk-free rate seems particularly important. However, even though a tractable one-factor model of the risk-free rate, such as the Cox et al. (1985) model, is assumed, the Fourier transform of the option value cannot be obtained in closed form, unless very strong restrictions are placed on the diffusion term of the spot price.

This study attempts to make the FT approach feasible under such models by employing an approximation to the characteristic function. Since the characteristic function is given basically by the (conditional) expectation, a method originally proposed by Shoji (2002) can be applied: conditional moments of diffusion processes are computed as the solution to a system of ordinary differential equations. Then, the computational burden does not increase drastically from that for some affine diffusion models in which the characteristic function is obtained by solving ordinary differential equations, known as the Riccati equations.

Numerical experiments are performed to examine the accuracy of the approximation, where benchmark option prices are obtained by the MC method. A model is considered that has CEV for both the spot price and volatility processes. The stochastic risk-free rate can be incorporated, which possibly correlates with both the spot price and volatility. For comparison purpose, an experimental setup regarding maturity length, moneyness, and parameter values is considered with reference to Hull and White (1987). The numerical results show that the third-order approximation generally achieves high accuracy, except the case where a volatility parameter of the volatility process is large. When mean reversion of the volatility process is relatively fast, the second-order approximation also exhibits a good performance.

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