Generalized fractional Ornstein-Uhlenbeck processes

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Abstract

An extended version of the fractional Ornstein-Uhlenbeck process (FOU) of which integrand is replaced by the exponential function of an independent Lévy process is considered. We call the process the generalized fractional Ornstein-Uhlenbeck process (GFOU). The process is also constructed by replacing the variable of integration of the generalized Ornstein-Uhlenbeck process (GOU) with an independent fractional Brownian motion (FBM). The stationary property and the auto-covariance function of the process are studied. Consequently, some conditions of stationarity and the long memory property of the process are obtained. Our underlying intention is to introduce the long memory property into the generalized fractional Ornstein-Uhlenbeck process which has the short memory property.

Keywords: Generalized Ornstein-Uhlenbeck processes, Lévy process, Stochastic integral, Long memory, Fractional Brownian motion.

1. Definition of GFOU.

We extend a Lévy process $\{\xi_t, t \geq 0\}$ into a two-side Lévy process $\{\xi_t, t \in \mathbb{R}\}$ by setting

$$\xi_t := \begin{cases} \xi_t^1 & \text{if} \quad t \geq 0 \\ -\xi_{(-t-)}^2 & \text{if} \quad t < 0, \end{cases}$$

where $\{\xi_t^1,\ t \geq 0\}$ and $\{\xi_t^2,\ t \geq 0\}$ are independent copies of $\{\xi_t, t \geq 0\}$. Throughout, assume a product probability space $(\Omega := \Omega_1 \times \Omega_2, \mathscr{F} := \mathscr{F}_1 \otimes \mathscr{F}_2, P := P_1 \otimes P_2)$. Let $\{\xi_t, t \in \mathbb{R}\}$ be a Lévy process on $(\Omega_1, \mathscr{F}_1, P_1)$ and $\{B_t^H,\ t \in \mathbb{R}\}$ be a FBM with index $H \in (0,1]$ on $(\Omega_2, \mathscr{F}_2, P_2)$, namely, both are independent. For $\lambda, \sigma > 0$, a generalized fractional Ornstein-Uhlenbeck process (GFOU) with initial value $x \in \mathbb{R}$ is defined as

$$Y_t^{H,x} := e^{-\lambda \xi_t} \left(x + \sigma \int_0^t e^{\lambda \xi_{s-}} dB_s^H \right).$$

If the initial variable satisfies (if definable) $\sigma \int_{-\infty+}^0 e^{\lambda \xi_s} dB_s^H$, we can write $Y_t^{H,x}$ as

$$Y_t^H := \sigma \int_{-\infty+}^t e^{-\lambda(\xi_t - \xi_{s-})} dB_s^H.$$

Throughout the paper we give results with brief explanations only. If you would like to go further into detail, please refer to Endo and Matsui (2007). The following notations are used throughout. Write $\overset{a.s.}{=}$ if equality holds almost surely. If the expectations only for a process $\{Z_t\}$ is considered, we write its expectation as E_Z.

2. Existence and Properties of GFOU

We do not investigate Y_t^x directly and investigate the stochastic integral

$$I_1^{a-b} := \int_a^b e^{-\lambda(\xi_b - \xi_{u-})} dB_u^H,\ \lambda \geq 0, \tag{1}$$

to cope with various situations. The results concerning the existence of integral are as follows.

Proposition 0.1 *Let $\{B_t^H,\ t \in \mathbb{R}\}$ be a FBM with $H \in (0,1]$ and $\{\xi_t,\ t \in \mathbb{R}\}$ be an independent two-sided Lévy process which is of finite variation. Then, there exists $\Omega_{01} \times \Omega_{02} := \Omega_0$ with $P(\Omega_0) = 1$ such that for each $\omega_1 \times \omega_2 \in \Omega_{01} \times \Omega_{02}$, the stochastic integral I_1^{a-b} with $-\infty < a < b < \infty$ defined in (1) exists finitely.*

Proposition 0.2 *Let $\{B_t^H,\ t \in \mathbb{R}\}$ be a FBM with $H \in (\frac{1}{2}, 1)$ and $\{\xi_t,\ t \in \mathbb{R}\}$ be an independent two-sided Lévy process. Suppose ξ_1 has the Laplace transform, namely, $E[e^{-\lambda\xi(1)}] < \infty$ for $\lambda > 0$ and $\lim_{t\to\infty} \overset{a.s.}{=} \infty$. Then, there exists $\Omega_{01} \times \Omega_{02} := \Omega_0$ with $P(\Omega_0) = 1$ such that for each $\omega_1 \times \omega_2 \in \Omega_{01} \times \Omega_{02}$ the stochastic integral I_1^{a-b} with $-\infty < a < b < \infty$ exists.*

Theorem 0.1 *Let $\{B_t^H,\ t \in \mathbb{R}\}$ be a FBM with $H \in (\frac{1}{2}, 1)$ and $\{\xi_t,\ t \in \mathbb{R}\}$ be an independent two-sided Lévy process. Suppose ξ_1 has the Laplace transform , namely, $E[e^{-\lambda\xi(1)}] < \infty$ for $\lambda > 0$ and $\lim_{t\to\infty} \overset{a.s.}{=} \infty$. Then, the stochastic integral I_1^{a-b} with $-\infty \le a < b < \infty$ exists. Furthermore, we can establish stochastic Fubini's theorem, i.e., for fixed $a < b$*

$$E\left[\int_a^b e^{-\lambda(\xi_b-\xi_{u-})} dB_u^H\right] = E_{B^H}\left[\int_a^b E_\xi\left[e^{-\lambda(\xi_b-\xi_{u-})}\right] dB_u^H\right] = 0, \tag{2}$$

and with $E[e^{-2\lambda\xi_1}] < 1$ for fixed $-\infty \le c < d < \infty$,

$$\begin{aligned} &E\left[\int_a^b e^{-\lambda(\xi_b-\xi_{u-})} dB_u^H \int_c^d e^{-\lambda(\xi_d-\xi_{v-})} dB_v^H\right] \\ &= E_{B^H}\left[\int_c^d \int_a^b E_\xi\left[e^{-\lambda(\xi_b-\xi_{u-}+\xi_d-\xi_{v-})}\right] dB_u^H dB_v^H\right] < \infty. \end{aligned} \tag{3}$$

The stationarity of Y_t^H is as follows.

Proposition 0.3 *The process Y_t^H is stationary under the condition of Theorem 0.1.*

3. Second Order Behavior of GFOU

We investigate the auto-covariance function of the GFOU process. Note that while the auto-covariance function of the GOU process decreases exponentially, that of the FOU process $\{Y_t^H\}$ with $\{\xi_t = t,\ a.s.\}$ decays like a power function.

We use the same notations as in Section 2 and obtain the following result.

Theorem 0.2 *Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ and $N = 0, 1, 2, \ldots$. Then, the GFOU process $\{Y_t^H\}$ based on a two-sided Lévy process $\{\xi_t, t \in \mathbb{R}\}$ and an independent FBM $\{B_t^H, t \in \mathbb{R}\}$ has the following asymptotic auto-covariance function under the assumptions of the existence of $I^{a-b}t_1$ in (1), the equations (2) and (3). For fixed $t \in \mathbb{R}$ as $s \to \infty$,*

$$\operatorname{Cov}\left(Y_t^H, Y_{t+s}^H\right) = \frac{1}{2}\sigma^2 \sum_{n=1}^{N} \theta_1^{-2n} \left(\prod_{k=0}^{2n-1} (2H-k)\right) s^{2H-2n} + O(s^{2H-2N-2}),$$

where $\theta_1 := -\log E[e^{-\lambda\xi_1}] > 0$.

4. Reference

[1] Endo, K. and Matsui, M. (2007) *Generalized fractional Ornstein-Uhlenbeck Processes* , Preprint.